Fourier Acoustics: Uncovering the Sources of Sound and Vibration*

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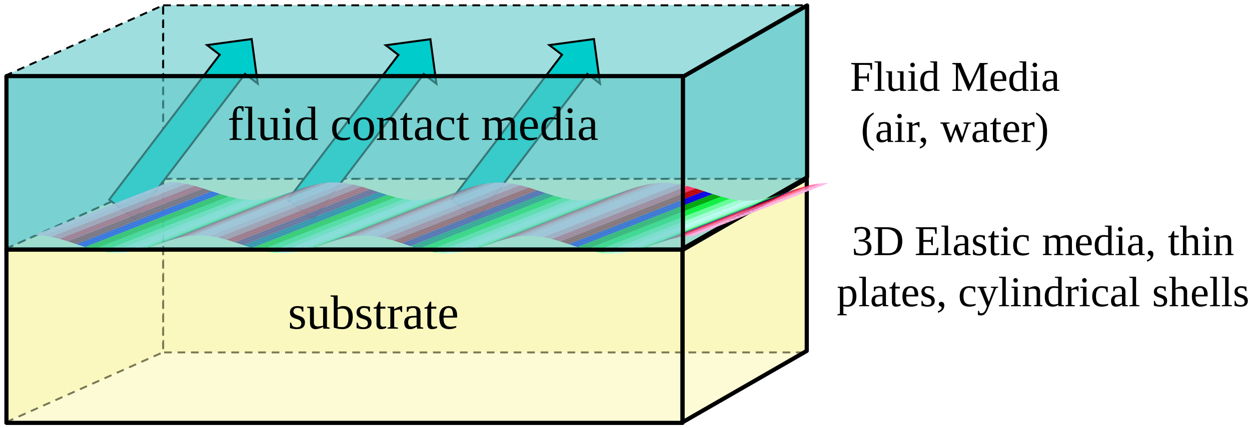
Overview of Material

- Consider the general problem – wave travelling in and on a substrate of infinite extent in contact with a fluid above.
  - Communication between the two media is the basis of our talk.
- Discuss methods to uncover the sources in the substrate that effectively couple to the contact media.
- Provide methods that expose the physics.



OUTLINE

- Slowness surface solutions (also called k-space, angular spectrum) to Helmholtz equations representing the media
- Coupling to the Fluid above
- Radiation to the Far-field
- Sources of radiation on interface
  - Concept of super-sonic imaging
  - Radiation from evanescent near-fields
- Slowness surface examples from Experiments at NRL
  - Towards understanding the physics of wave propagation in the substrate - Beaming
- Dispersion Space – Bandgaps
  - Application to a metamaterial

Some Relevant Helmholtz equations ($e^{-i\omega t}$ time tag) in an Isotropic Infinite Media

- **Electromagnetics:**

$\nabla^2 \vec{E}(x,y,z) + \frac{\omega^2}{c^2}\vec{E}(x,y,z) = 0$ where $\vec{E} = \{E_x, E_y, E_z\}$ and $c = \sqrt{1/\epsilon\mu}$.

- **Acoustics:**

$\nabla^2 p(x,y,z,\omega) + \frac{\omega^2}{c^2} p(x,y,z,\omega) = 0$ where p =pressure and $c = \sqrt{B/\rho}$.

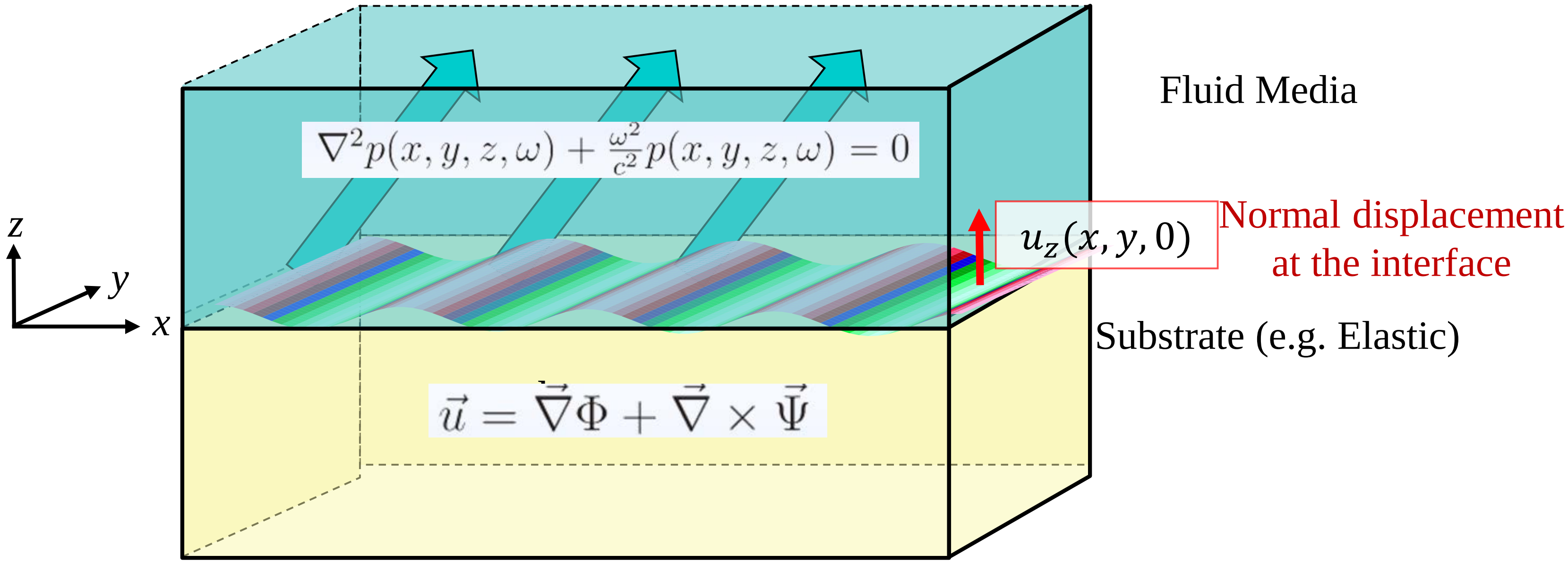
- **Elastic Solids:**

P waves: $\nabla^2 \Phi + \frac{\omega^2}{c_p^2}\Phi = 0$ where $c_p = \sqrt{(\lambda + 2\mu)/\rho}$

S waves: $\nabla^2 \vec{\Psi} + \frac{\omega^2}{c_s^2}\vec{\Psi} = 0$ where $c_s = \sqrt{\mu/\rho}$. Rayleigh Surface waves:

Displacement is $\vec{u} = \vec{\nabla}\Phi + \vec{\nabla} \times \vec{\Psi}$

Helmholtz Equations in Our Problem



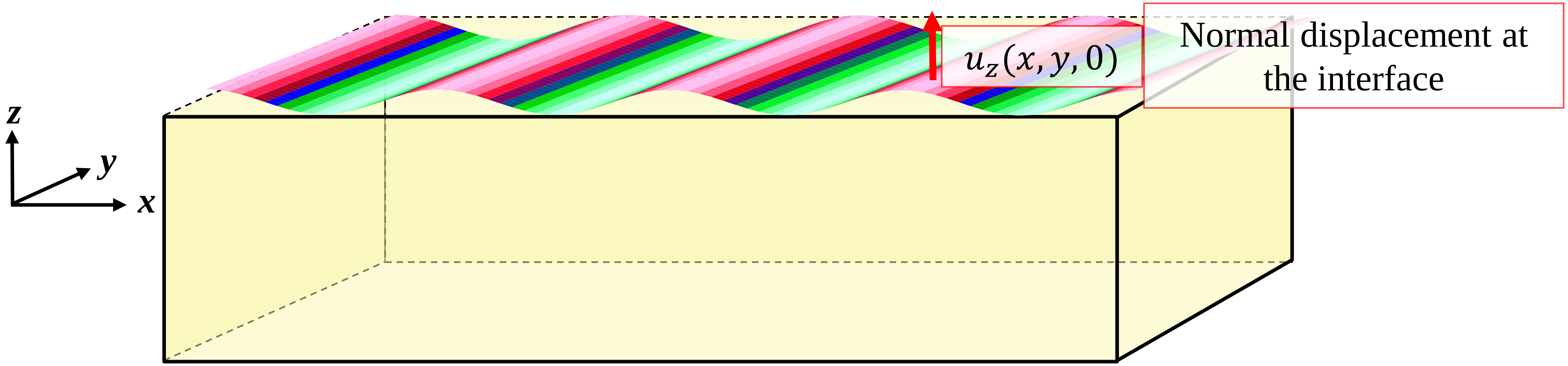
A Simple solution that satisfies the Helmholtz Equations at the interface is a Monochromatic Wave given by $u_z = e^{ik_x x + ik_y y - i\omega t}$

We study this solution in detail

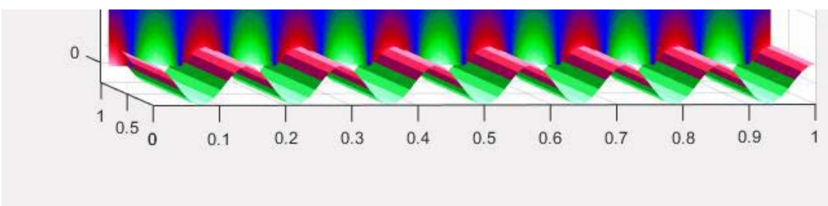
Details of the Monochromatic Wave Solution: Slowness Space

A **monchromatic wave solution** is $u_z = e^{ik_x x} e^{ik_y y} e^{-i\omega t}$ in a plane $z = 0$.

If we further define $k_x = \omega / c_x$, $k_y = \omega / c_y$, solutions look like $e^{i\omega(x/c_x + y/c_y - t)}$. For example, when $k_y = 0$ we have $e^{i\omega(x - c_x t)/c_x}$ representing a phase front (wave) travelling to the right with the phase speed c_x.



right-going wave $c_x > 0$

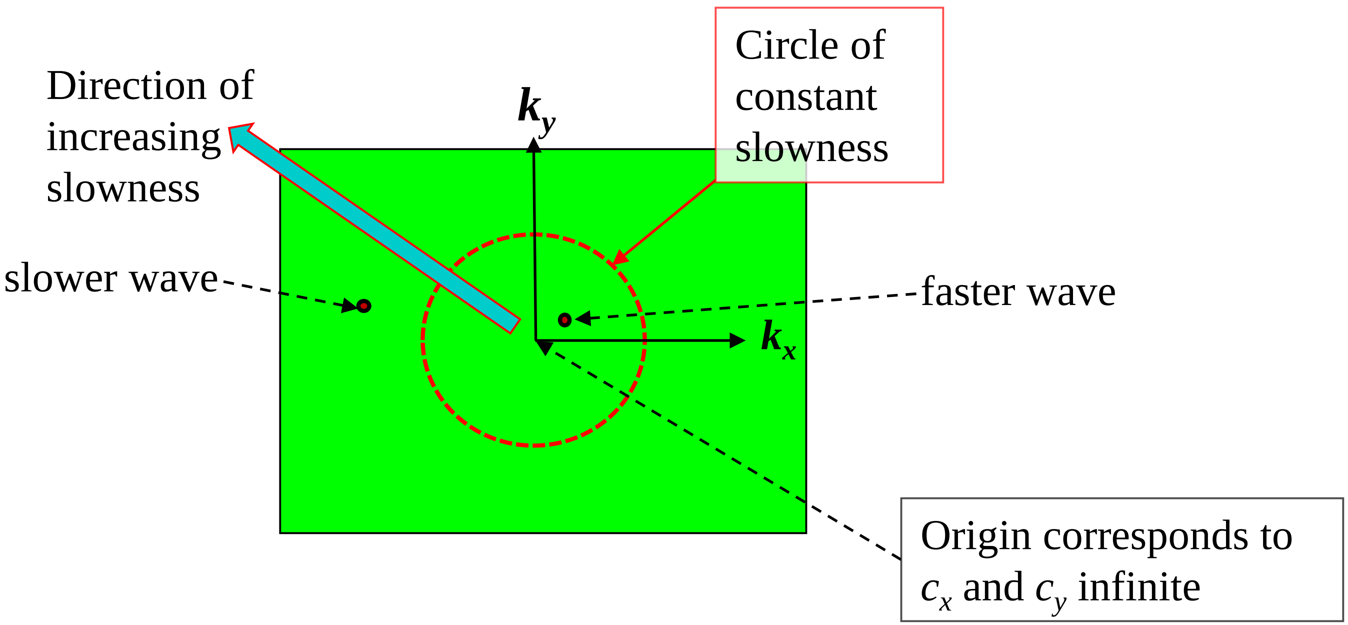


Note: For a left-going wave we put $c_x < 0$

- Since $k_x \propto \frac{1}{c_x}$ it represents a slowness (m/s) and we can view (k_x, k_y) as a SLOWNESS SPACE and (c_x, c_y) the speed of the Monochromatic wave in the coordinate directions.

The Slowness (k-space) Space

$$k_x = \omega / c_x,\ k_y = \omega / c_y$$

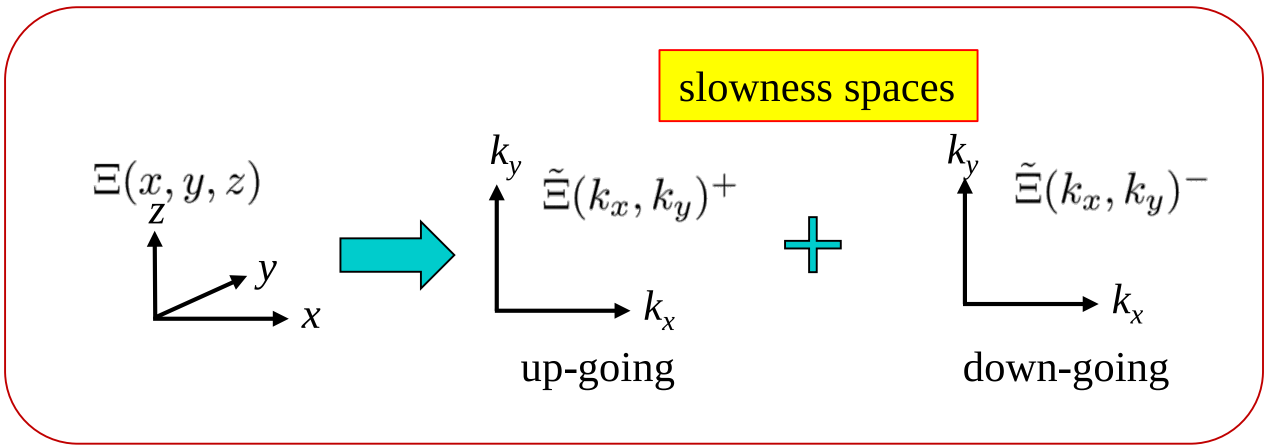


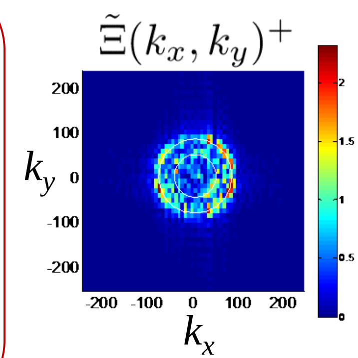
We represent spatial field as a sum of Monochromatic Wave Solutions – Two Slowness Spaces

- In general the solution of these Helmholtz equations can be written, with $\Xi \equiv \{p, \Psi, \Phi, E_x, E_y, E_z, B_x, B_y, B_z, \cdots\}$, as a 2D Inverse Fourier transform

$$\Xi(x,y,z) = \frac{1}{4\pi^2}\iint_{-\infty}^{\infty} e^{ik_x x} e^{ik_y y}\left[\underbrace{\tilde{\Xi}(k_x,k_y)^+ e^{ik_z z}}_{\text{up-going}} + \underbrace{\tilde{\Xi}(k_x,k_y)^- e^{-ik_z z}}_{\text{down-going}}\right] dk_x dk_y$$

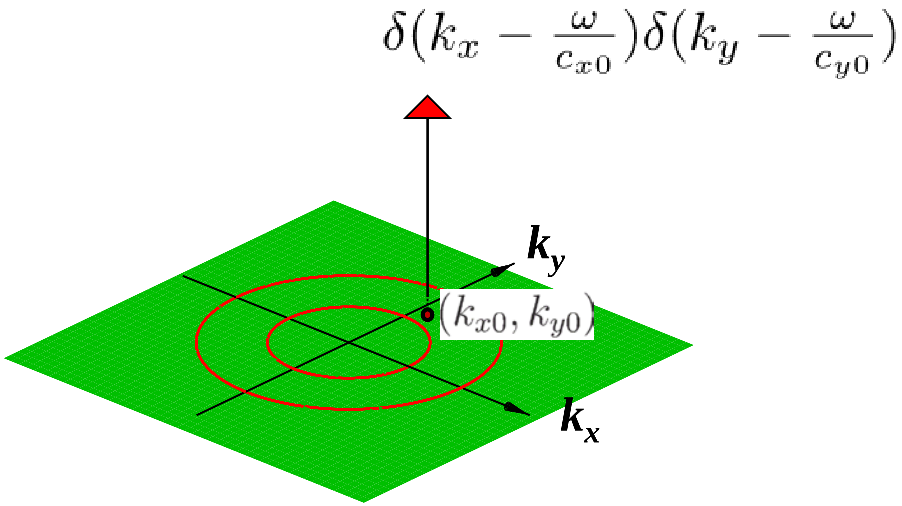
where $\tilde{\Xi}(k_x,k_y)^{\pm}$ are the slowness space field amplitudes and $k_z = \sqrt{\frac{\omega^2}{c_\xi^2} - (k_x^2 + k_y^2)}$.





The Amplitude of a Monochromatic Wave $e^{ik_{x0}x}e^{ik_{y0}y}e^{-i\omega t}$ is represented by a delta function in slowness space

$$\tilde{\Xi}(k_x, k_y)^{\pm} = \delta(k_x - k_{x0})\delta(k_y - k_{y0})$$

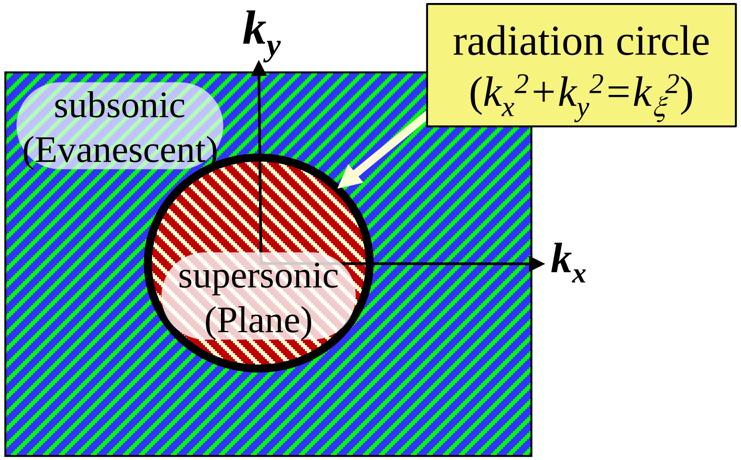


The Radiation Circle

- Consider up-going term $\tilde{\Xi}(k_x, k_y)^+ e^{ik_z z}$ where we had $k_z = \sqrt{\frac{\omega^2}{c_\xi^2} - (k_x^2 + k_y^2)}$. The circle defined by the argument of the square root, $(k_x^2 + k_y^2) = \frac{\omega^2}{c_\xi^2}$ defines the break point between up-going propagating wave and an up-going evanescent wave. We call this circle the **radiation circle**.

- Inside radiation circle $(k_x^2 + k_y^2) < k_\xi^2$, so k_z is real, $e^{ik_z z}$ is a **phase change**, no amplitude change $(k_\xi \equiv \omega / c_\xi)$

- Outside radiation circle $(k_x^2 + k_y^2) > k_\xi^2$, so k_z is pure imaginary and $e^{ik_z z} = e^{-|k_z| z}$, an **evanescent decay**



- Inside radiation circle is faster than c_ξ so we call the monochromatic waves **supersonic**
- Outside radiation circle is slower than c_ξ so we call the monochromatic waves **subsonic**

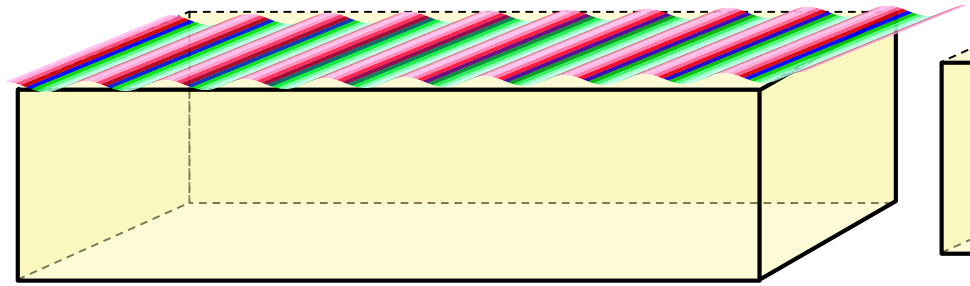
Who does this decomposition?

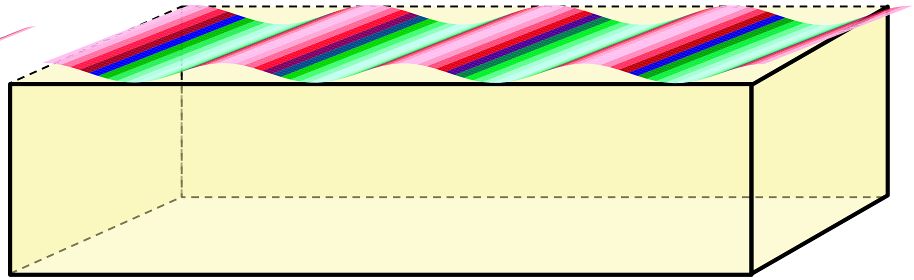
1) In General: Works involving Layered Media involving Electromagnetic, Elastic, Seismic or Fluid waves.
2) Weng Cho Chew, *Waves and Fields in Inhomogeneous Media*, (IEEE Press, New York,1995).
3) R. W. P. King, *Lateral Electromagnetic Waves; Theory and Applications to Communications, Geophysical Exploration, and Remote Sensing* (Springer-Verlag, New York, 1992).
4) C. P. A. Wapenaar and A. J. Berkhout, *Elastic Wave Field Extrapolation*, (Elsevier, 1989).
5) Earl G. Williams, *Fourier Acoustics, Sound radiation and Near-field Acoustical Holography*, (Academic Press, London 1999).

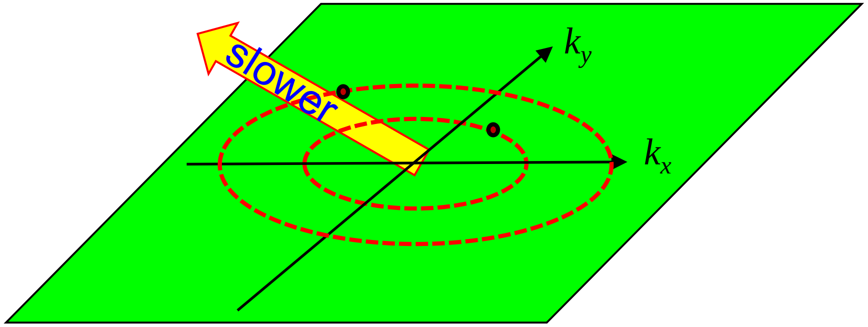
TEST QUESTION #1

Which is the slower monochromatic wave, left or right picture?

hint: $c_x = \lambda_x f$







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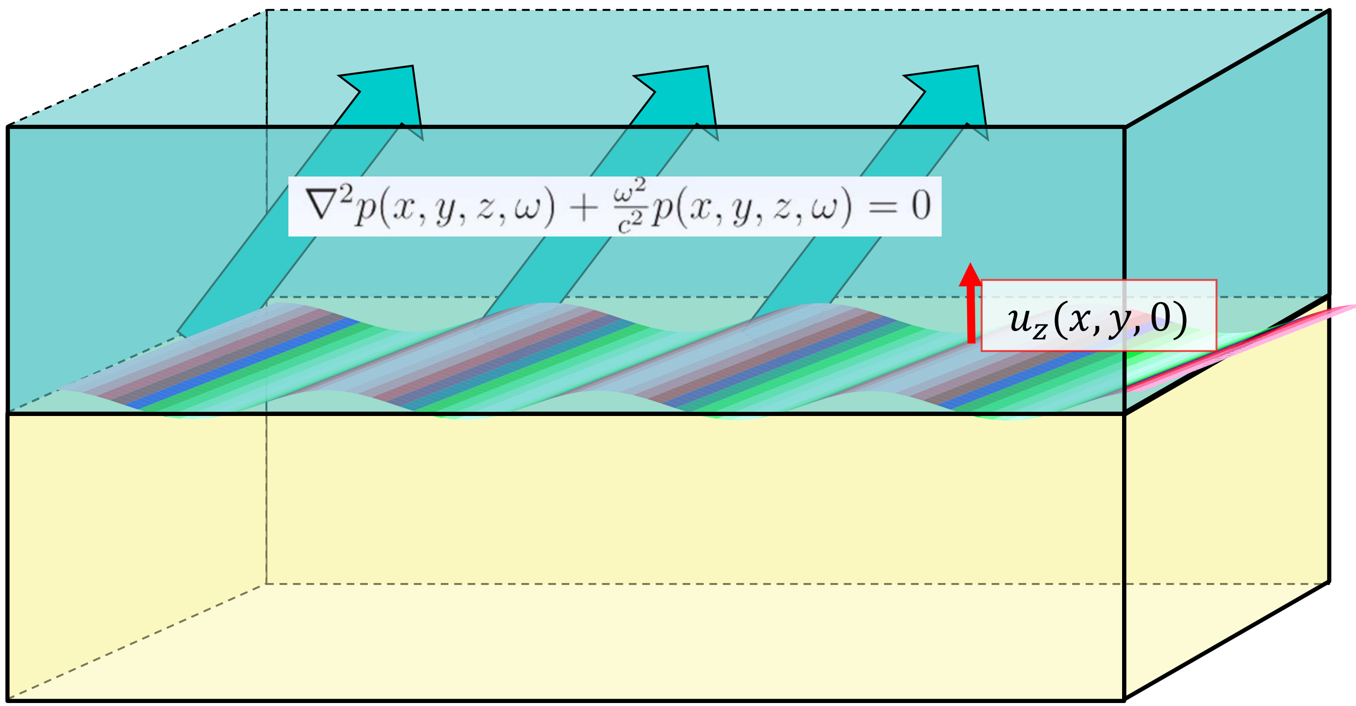
Consider Radiation into the fluid above

Assume no down-going waves in fluid media

$$p(x,y,z) = \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} e^{ik_x x} e^{ik_y y} \left[\underbrace{P(k_x,k_y)^{+} e^{ik_z z}}_{\text{up-going}} + \underbrace{P(k_x,k_y)^{-} e^{-ik_z z}}_{\text{down-going}} \right] dk_x dk_y$$



How does substrate communicate with the fluid? Mathematically via the Rayleigh Integral

Whatever the wave type, only normal surface velocity $v_n = -i\omega u_z$ is needed to determine uniquely the pressure in the fluid media

- Planar Geometry we have Rayleigh's first integral formula

$$p(\mathbf{r}') = \frac{-i\omega\rho}{2\pi}\int_{S_r}\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}v_n(\mathbf{r})dx\,dy$$

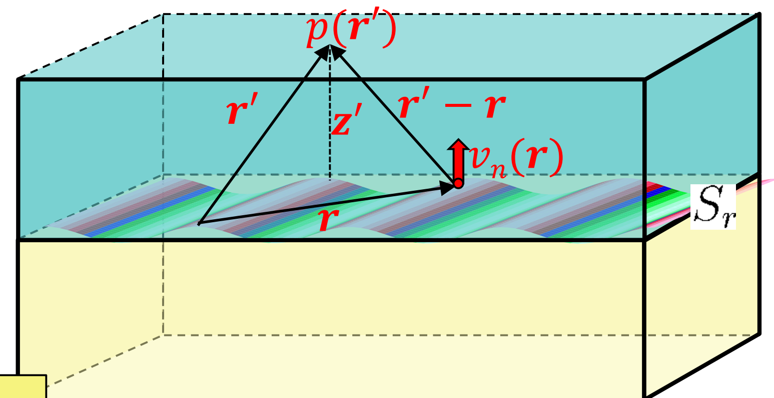
$k = \omega/c_\xi = \omega/c$

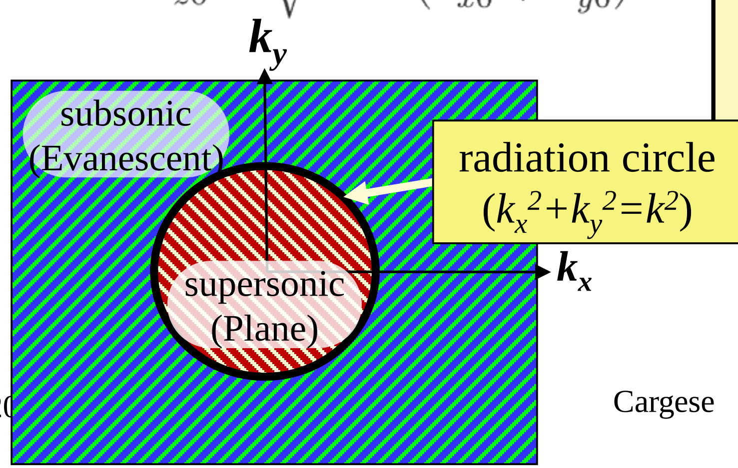
- Example: Assume again monochromatic wave $v_n(\mathbf{r}) = v_0 e^{ik_{x0}x}e^{ik_{y0}y}$ in the plane at $z=0$ then the solution from the Rayleigh integral is

$$p(\mathbf{r}') = \frac{\omega\rho_0}{k_{z0}}e^{ik_{z0}z'}\left(v_0 e^{ik_{x0}x'}e^{ik_{y0}y'}\right)$$

up-going

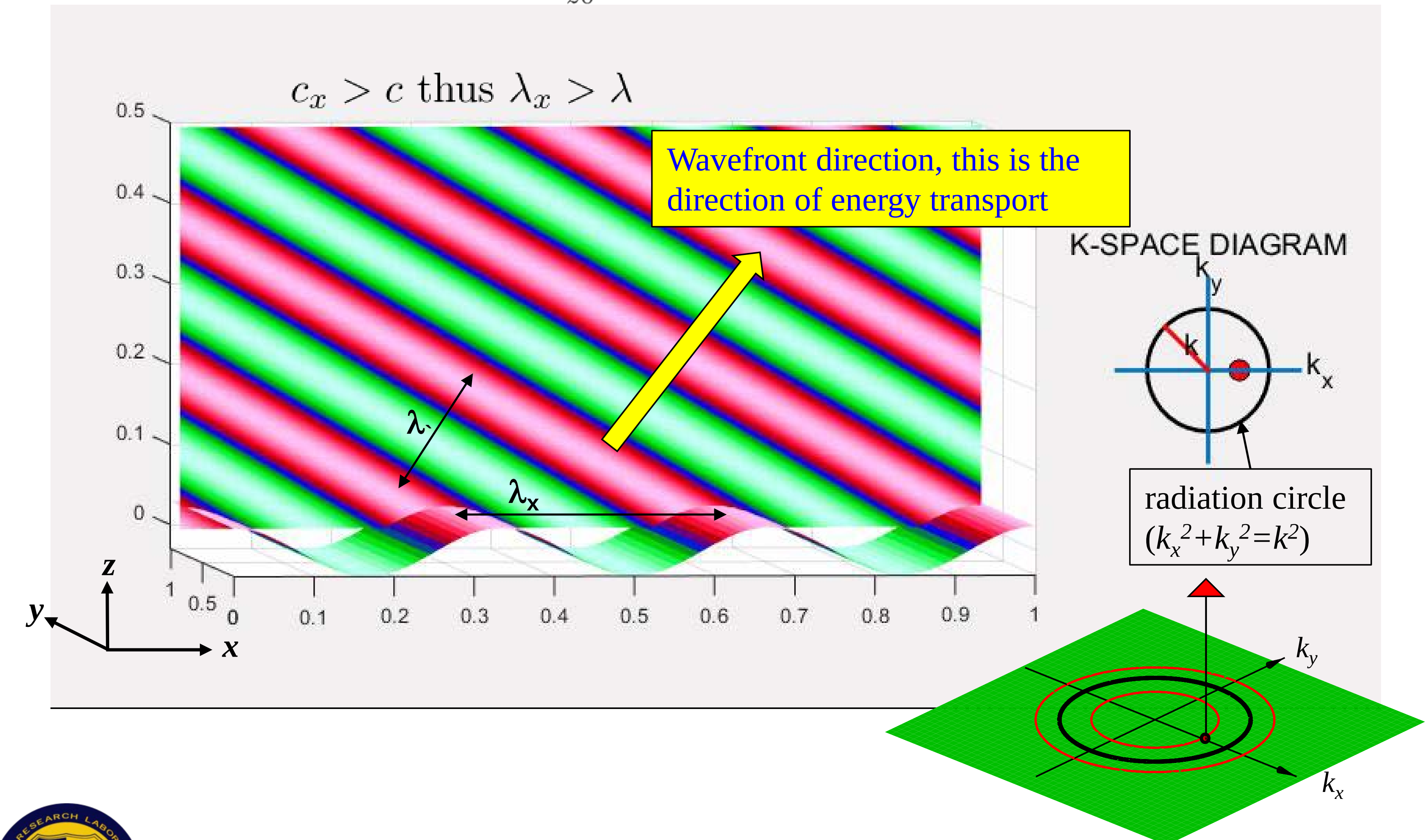
where $k_{z0} = \sqrt{k^2 - (k_{x0}^2 + k_{y0}^2)}$





The SuperSonic Wave

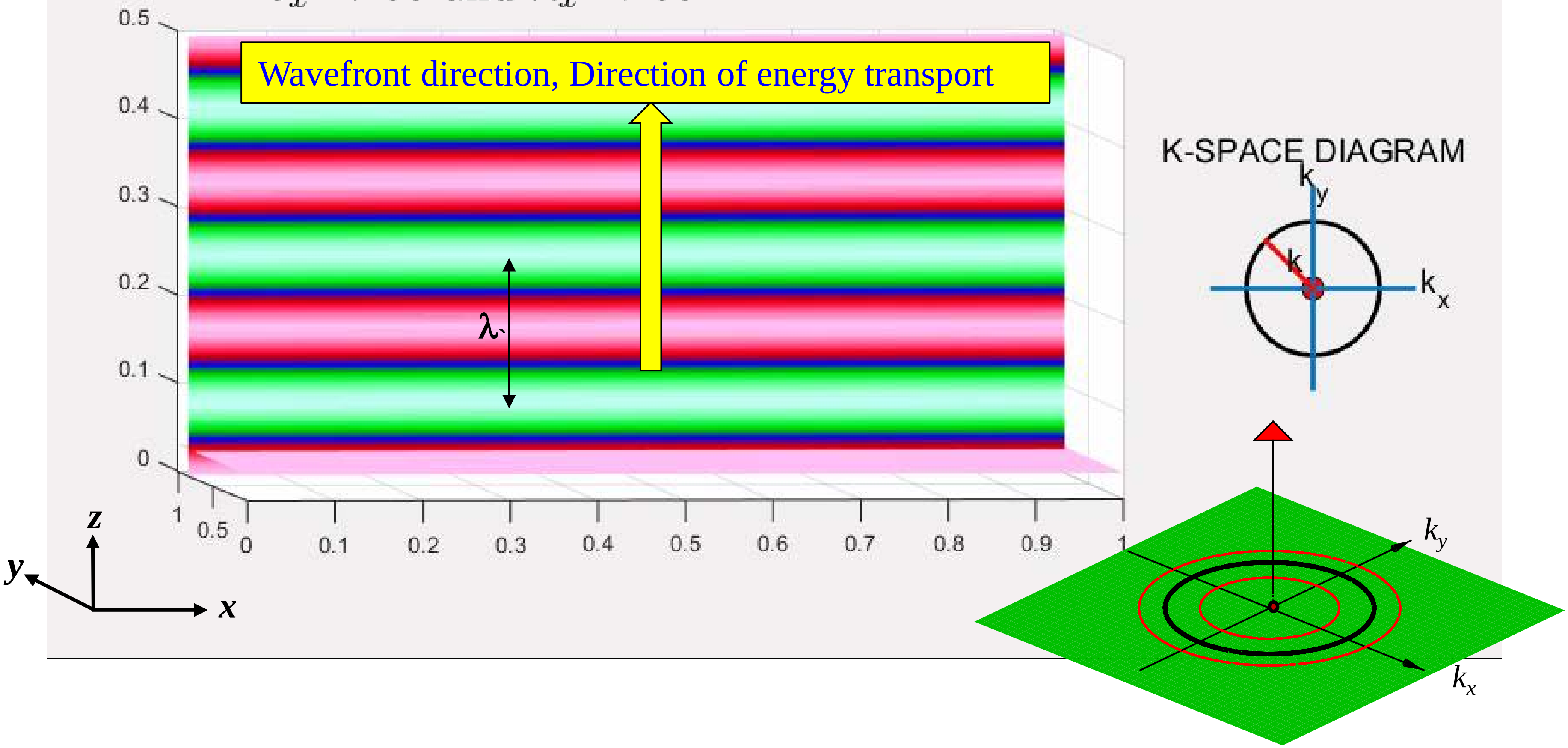
$$p(x,y,z) = \left(v_0 e^{ik_{x0}x} e^{ik_{y0}y}\right) \frac{\omega\rho_0}{k_{z0}} e^{ik_{z0}z}, \quad \text{with} \quad k_{x0}^2 + k_{y0}^2 < k^2, \; k_{z0} \quad \text{is real.}$$



The SuperSonic Wave of Infinite Speed

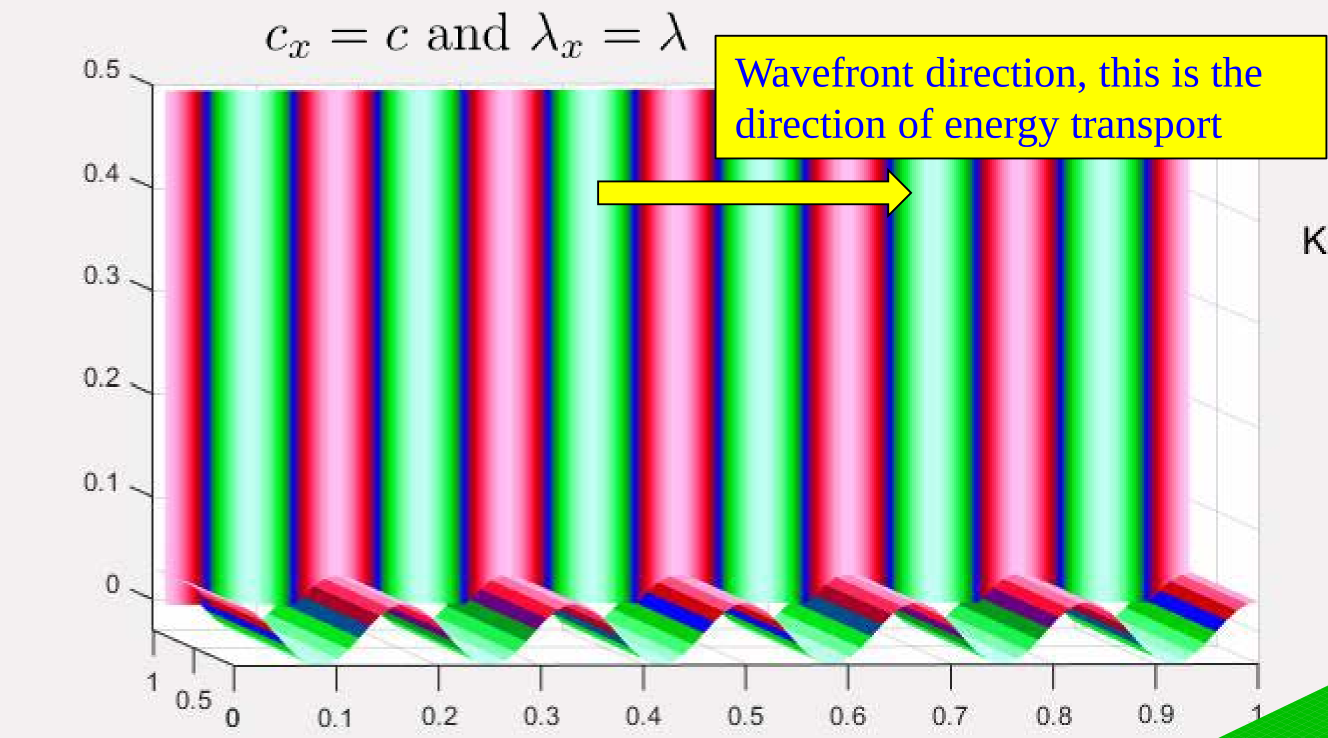
$$k_{x0} = 0, k_{y0} = 0, k_{z0} = k$$

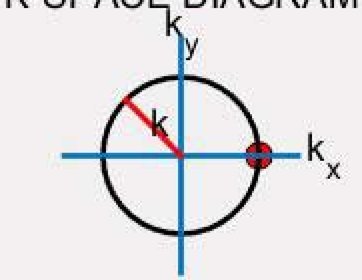
$$c_x \to \infty \text{ and } \lambda_x \to \infty$$

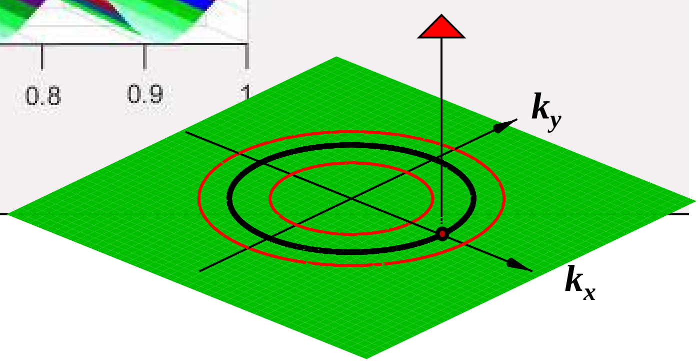


The SuperSonic Wave on the Radiation Circle

$$p(x,y,z) = \left(v_0 e^{ik_{x0}x} e^{ik_{y0}y}\right) \frac{\omega \rho_0}{k_{z0}}, \quad \text{with} \quad k_{x0}^2 + k_{y0}^2 = k^2, \; k_{z0} = 0.$$



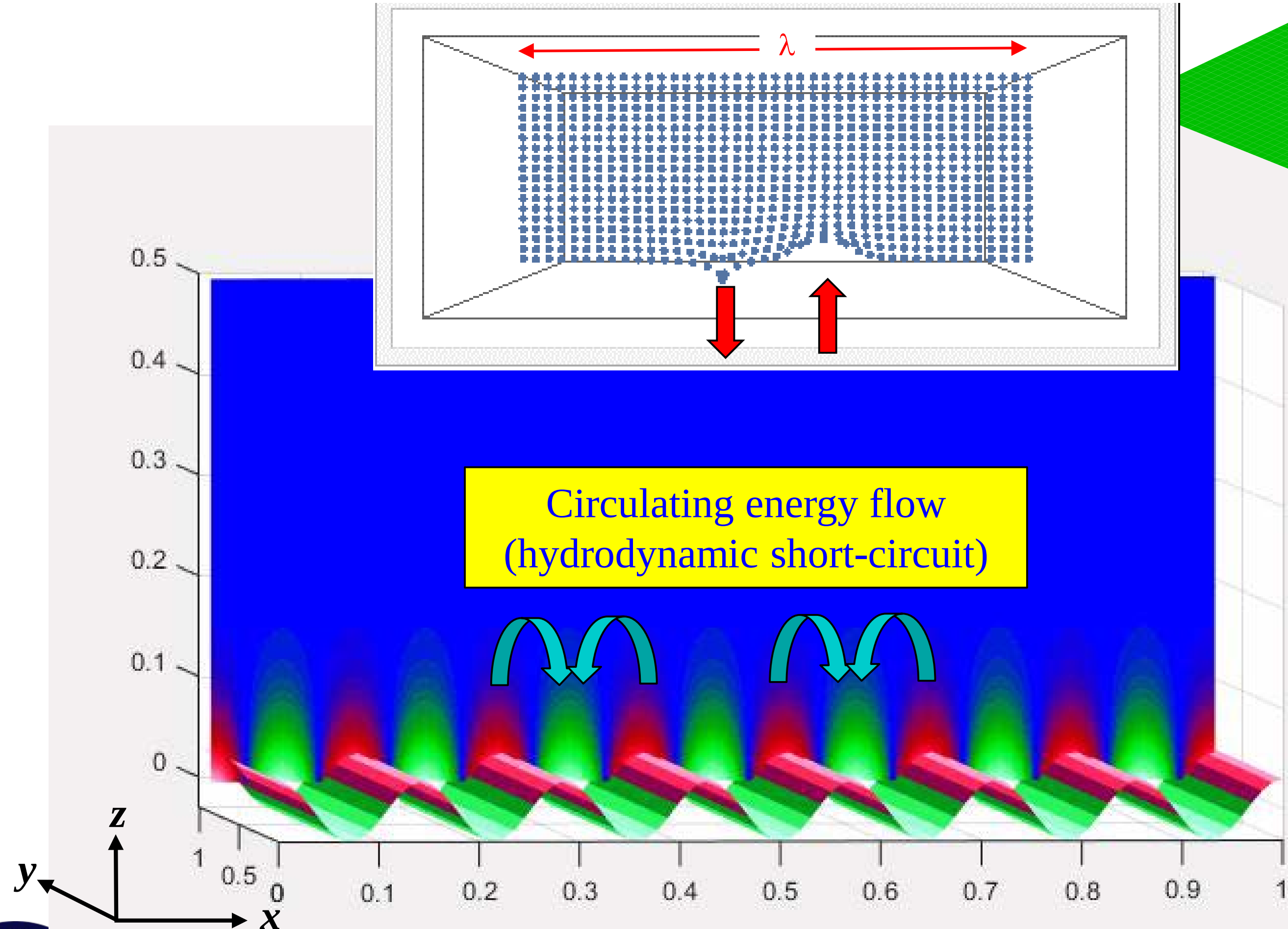


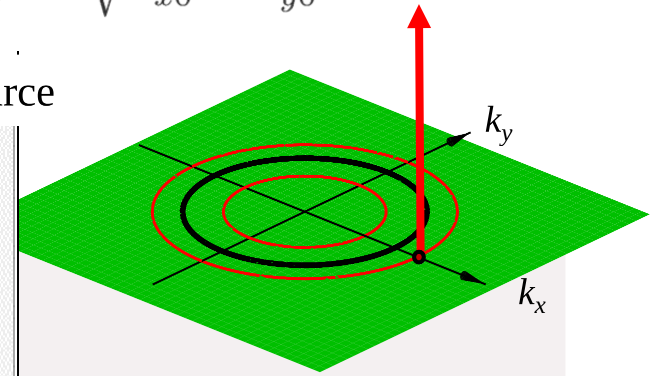


The Subsonic Wave: Evanescent Decay

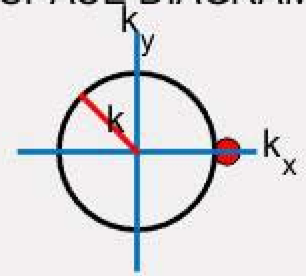
$$p(x,y,z) = \left(v_0 e^{ik_{x0}x} e^{ik_{y0}y}\right) \frac{\omega\rho_0}{k_{z0}} e^{-|k_{z0}|z},\ k_{x0}^2 + k_{y0}^2 > k^2,\ k_{z0} = i\sqrt{k_{x0}^2 + k_{y0}^2 - k^2}.$$

Particle motion in fluid above a dipole volume source



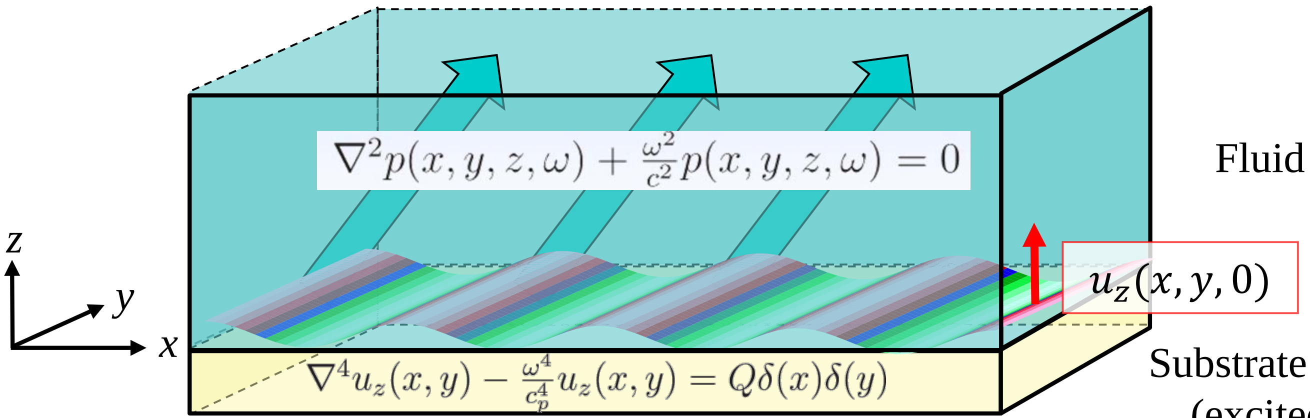


K-SPACE DIAGRAM



$c_x < c$ and $\lambda_x < \lambda$

Substrate is a thin Elastic Plate

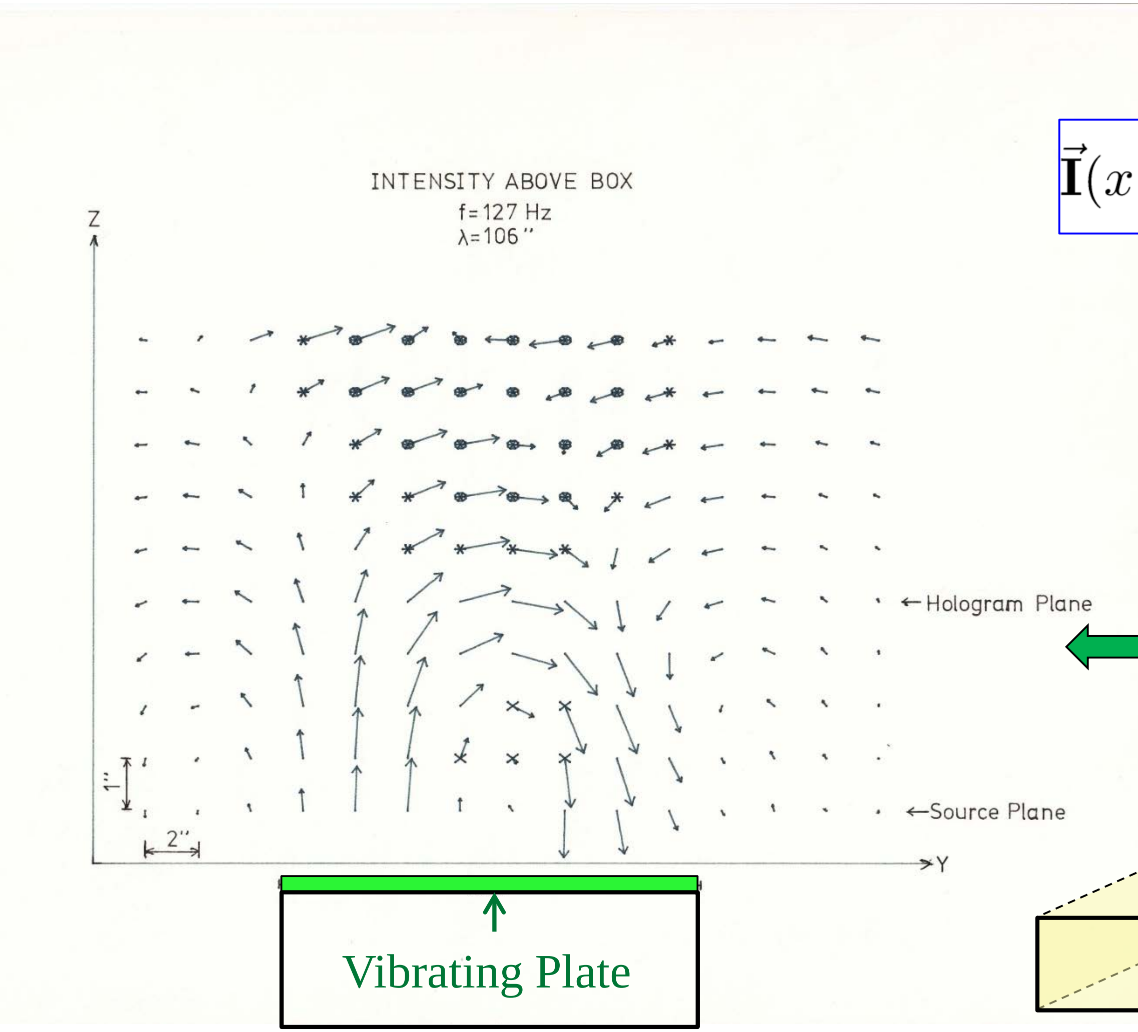


Fluid Media

Normal displacement
at the interface

Substrate is a thin elastic plate
(excited with point force)

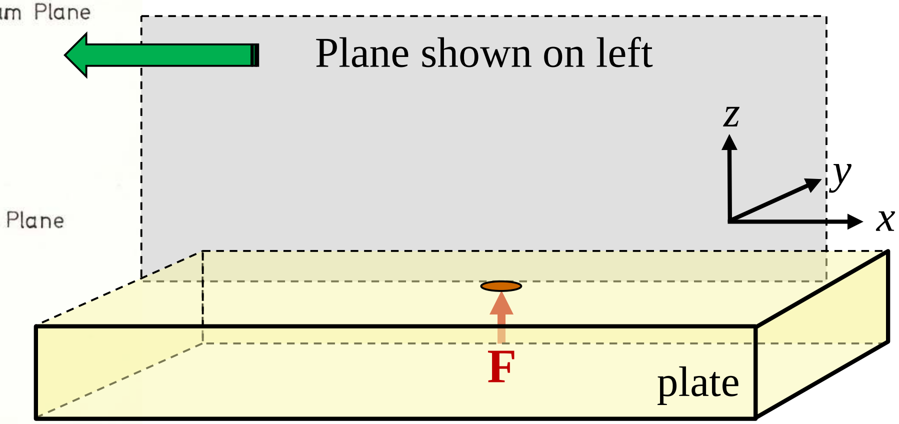
Circulating Energy paths due to evanescent waves: Experiment with a vibrating plate substrate



$$\vec{\mathbf{I}}(x,y,z) = \frac{1}{2}\Re[p(x,y,z)\vec{\mathbf{v}}(x,y,z)^*]$$

$$\text{Power} = \int_S \vec{\mathbf{I}}(x,y,z_0)\cdot\hat{\mathbf{n}}\;dx\,dy$$

$$z_0 = \text{const.}$$



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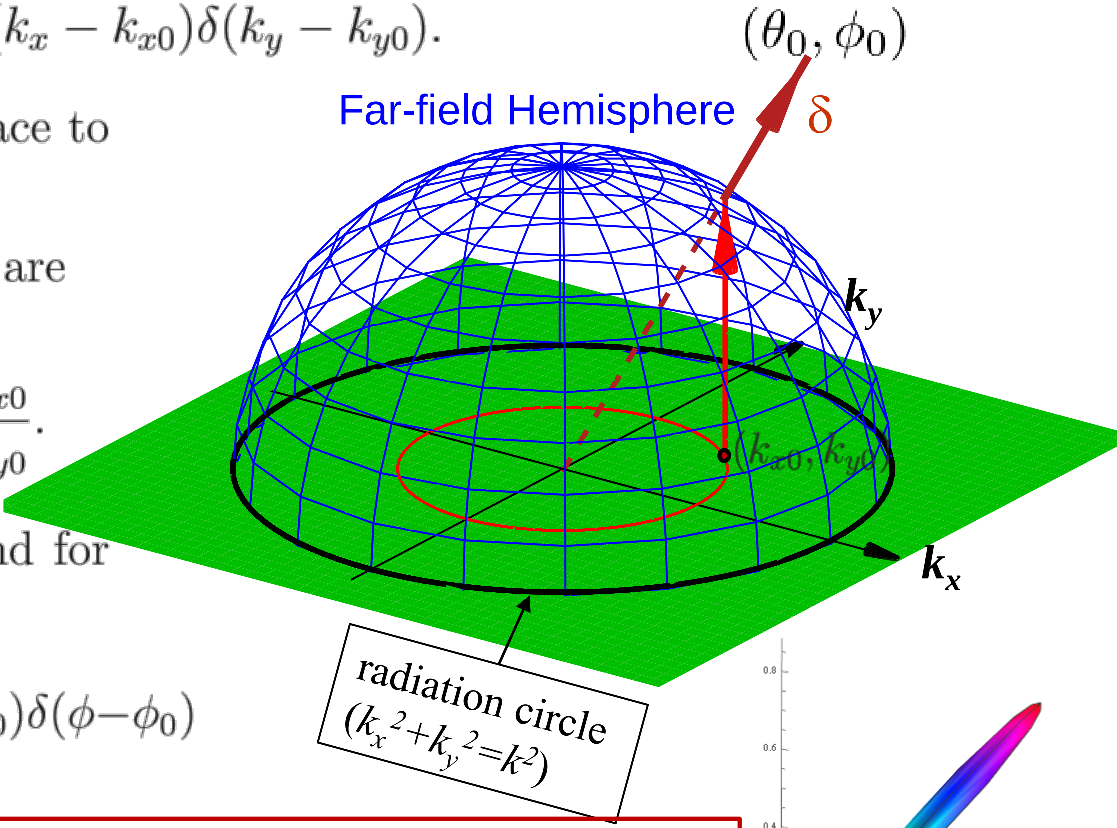
Slowness Space and the Far Field

The directivity pattern in the far-field is given by slowness space amplitudes inside the radiation circle (with a simple transformation rule)

We saw that $\mathcal{F}_{xy}[e^{ik_{x0}x}e^{ik_{y0}y}] \rightarrow \delta(k_x - k_{x0})\delta(k_y - k_{y0})$.



- Transformation Rule, slowness space to FF hemisphere (spherical coords):

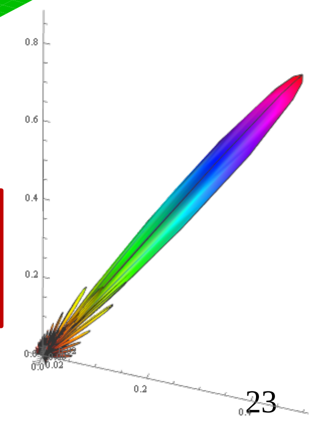
The spherical angles for the far-field are

$$\sin\theta_0 = \frac{\sqrt{k_{x0}^2 + k_{y0}^2}}{k}, \quad \tan\phi_0 = \frac{k_{x0}}{k_{y0}}.$$

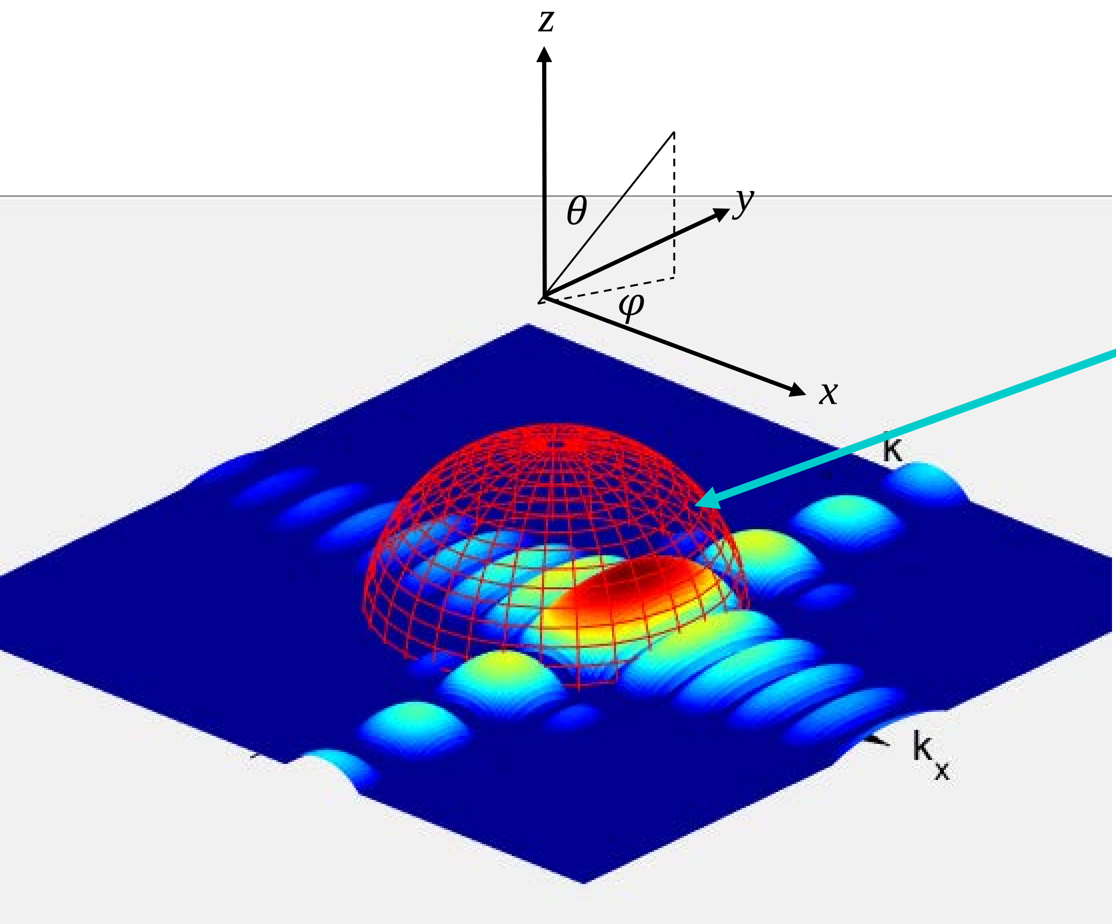
With this transformation rule we find for a monochromatic wave

$$\delta(k_x - k_{x0})\delta(k_y - k_{y0}) \rightarrow \delta(\cos\theta - \cos\theta_0)\delta(\phi - \phi_0)$$

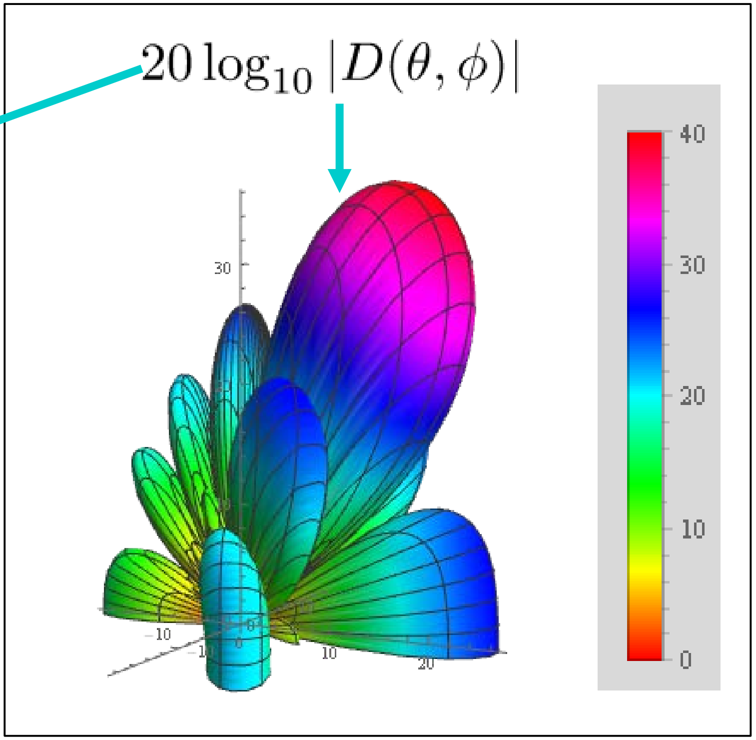
Simply put: Points within radiation circle are projected vertically upwards until they intersect the radiation hemisphere



Example of the Projection to Far-Field



2D Fourier Transform of Surface Velocity



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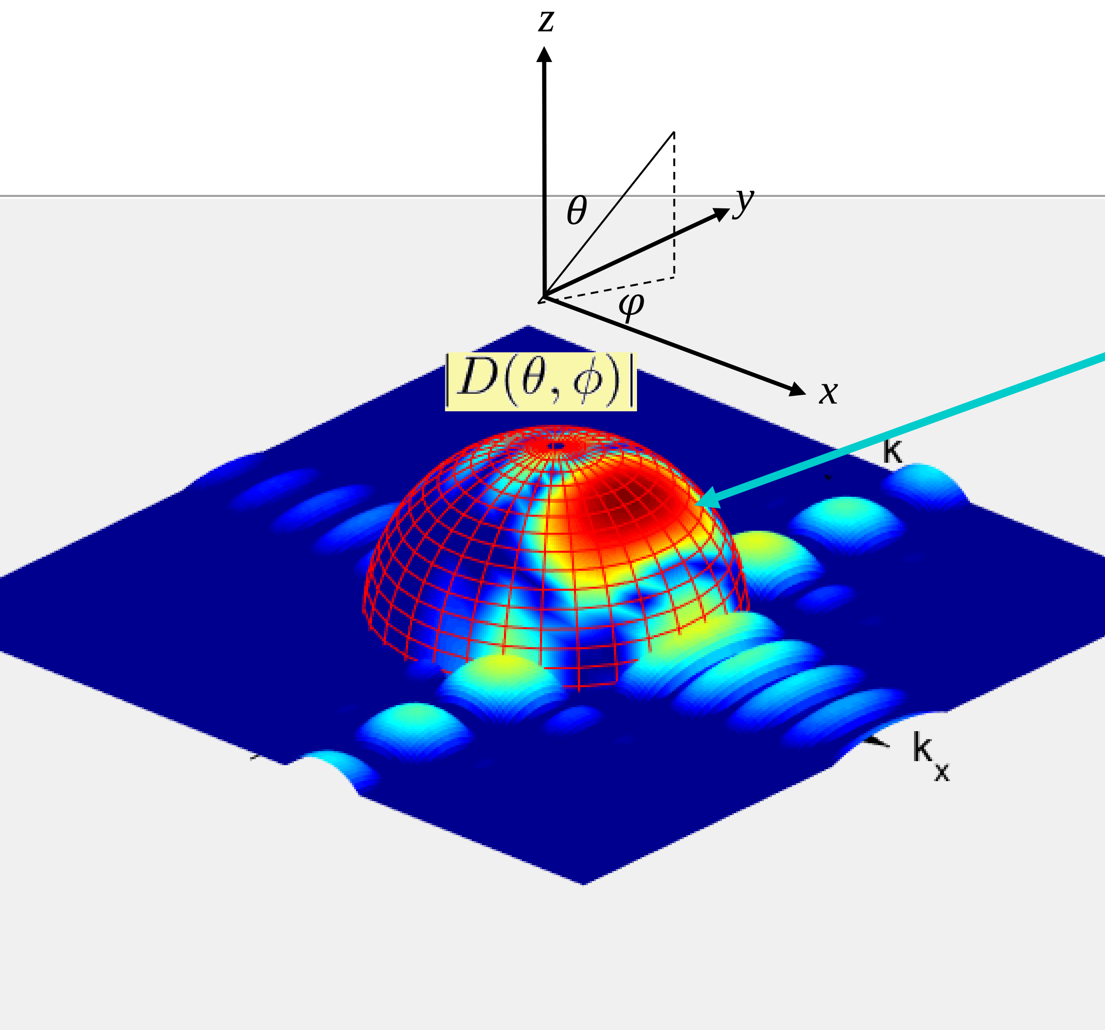
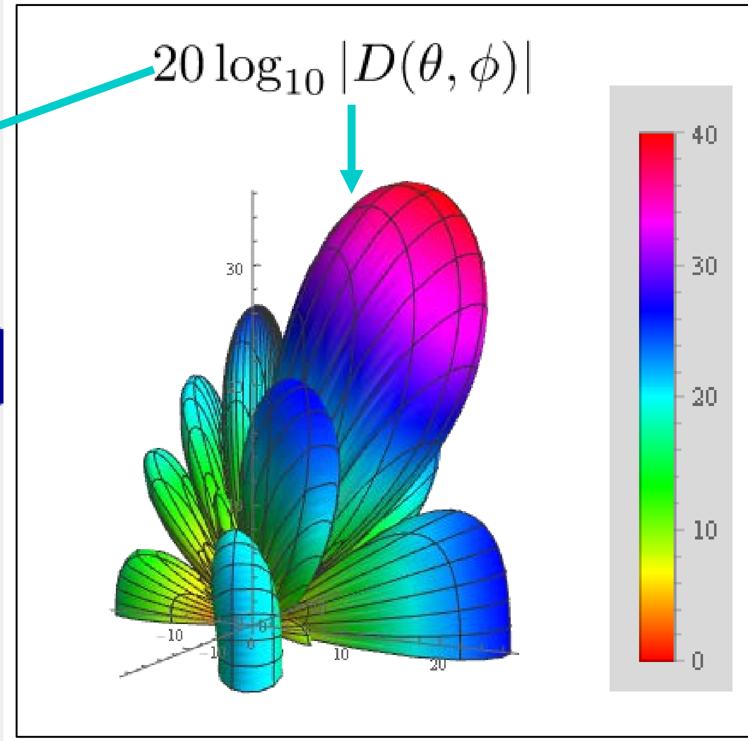
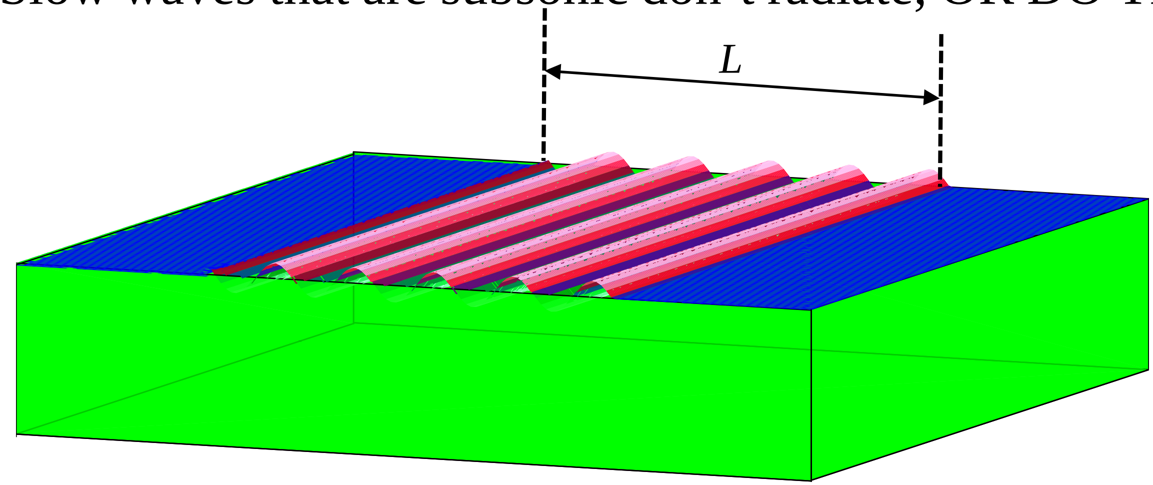
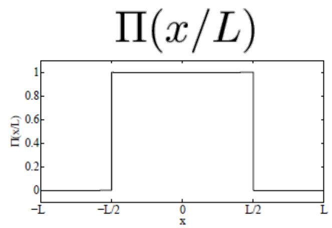




Finite Media Effects in Slowness Space

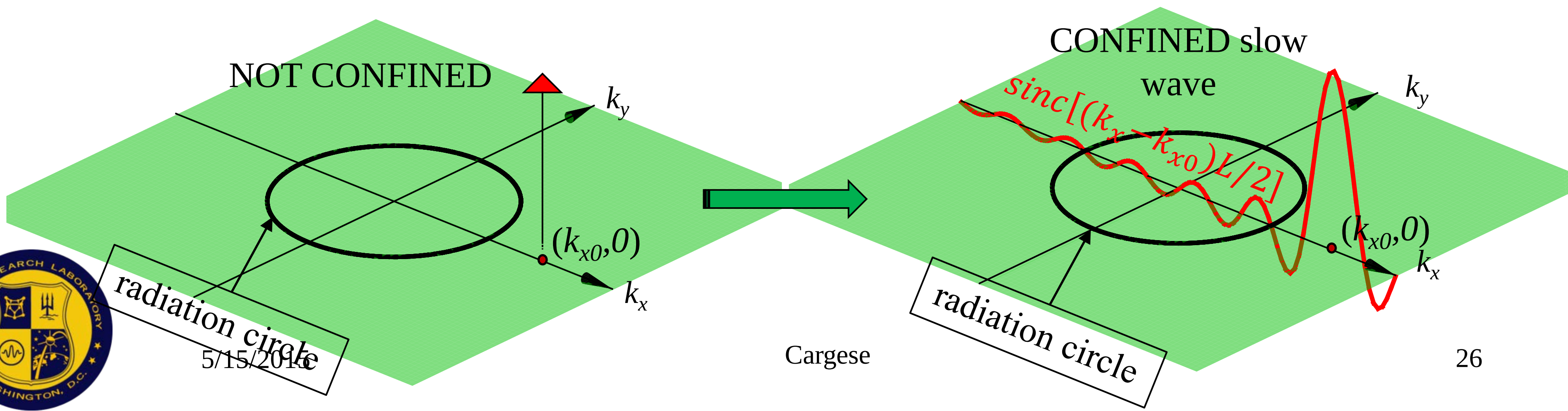
Slow waves that are subsonic don't radiate, OR DO THEY?



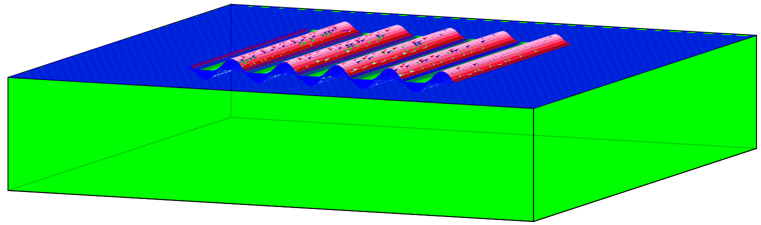


- Recall the spatial Fourier transform of a window $\Pi(x/L)$ is $\mathcal{F}_x[\Pi(x/L)] = L\mathrm{sinc}(k_x L/2)$. Thus the FT of a confined wave is

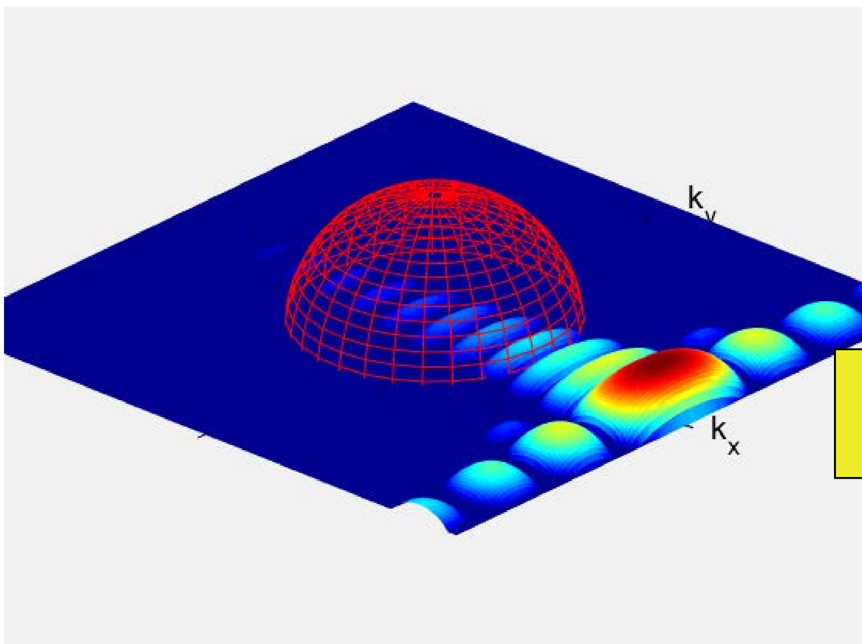
$$\mathcal{F}_x[e^{ik_{x0}x}\Pi(x/L)] = \mathcal{F}_x[e^{ik_{x0}x}]\otimes\mathcal{F}_x[\Pi(x/L)] = L\mathrm{sinc}[(k_x - k_{x0})L/2]$$



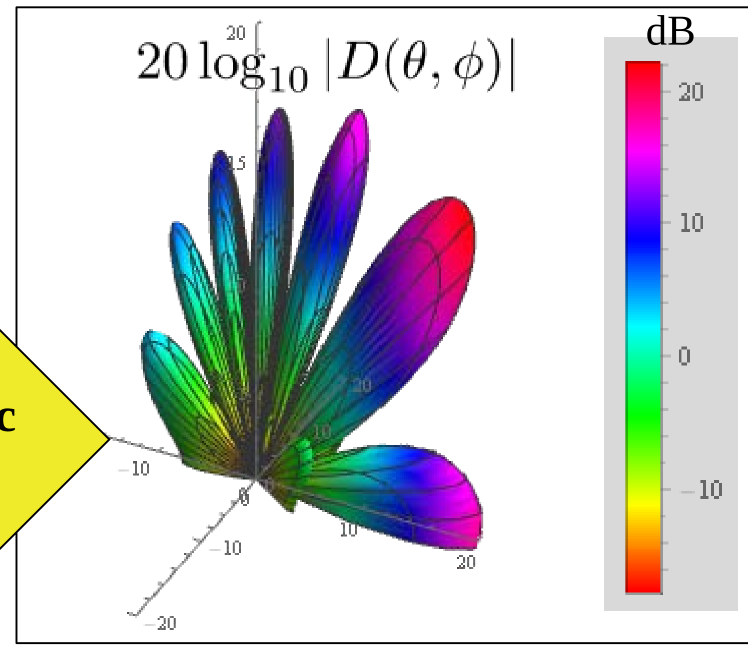
Example: wave confined in both directions
Far-field is a product of sinc functions



Fourier Transform is $\mathrm{sinc}[(k_x - k_{x0})L/2]\ \mathrm{sinc}[k_y L_y/2]$



Subsonic wave



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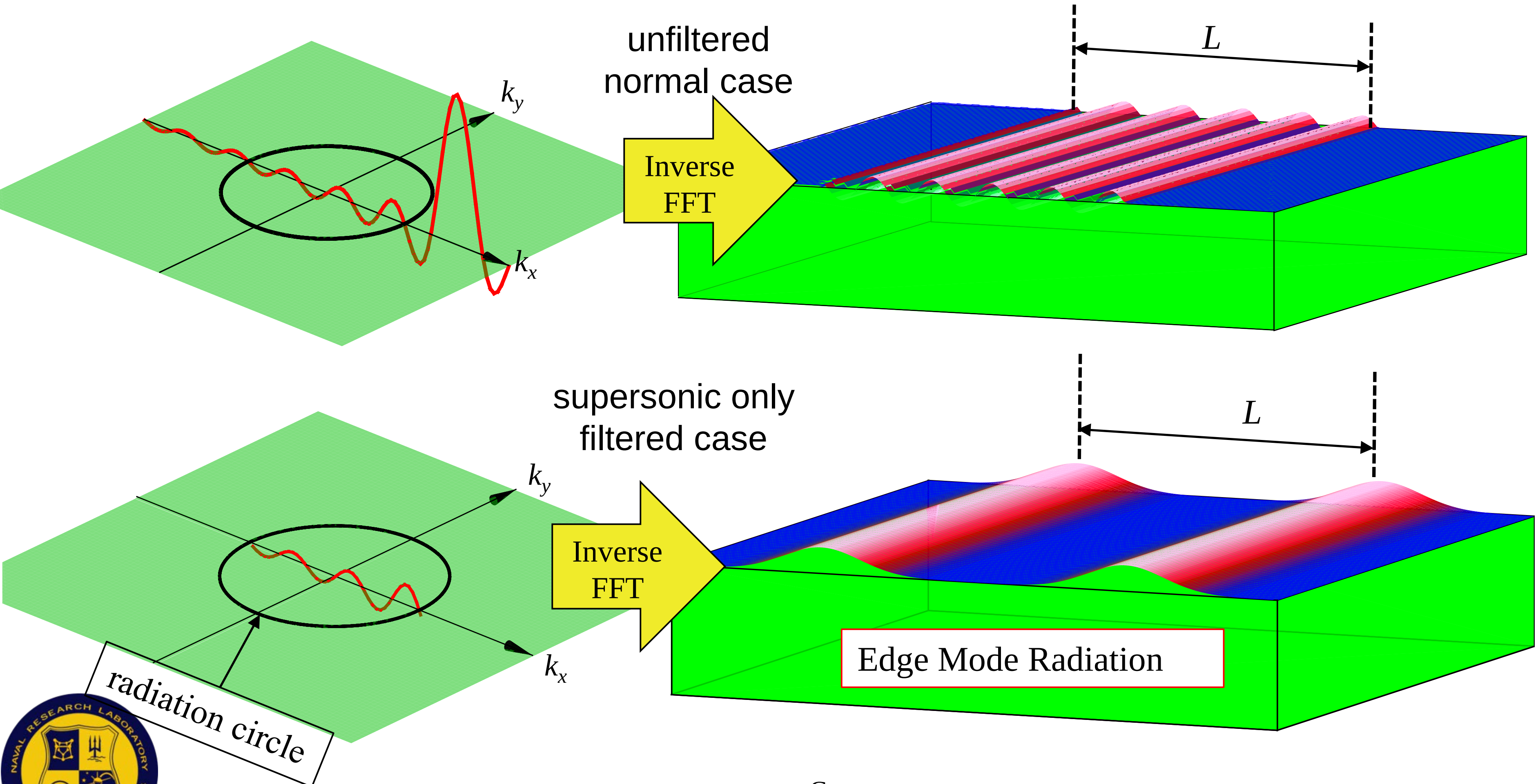
But where spatially does the radiation come from?

- (Supersonic imaging): what if we inverse transform back to real space, *but use only the data in the radiation circle*? We know that whatever that spatial field is – it is what radiates to the far-field.

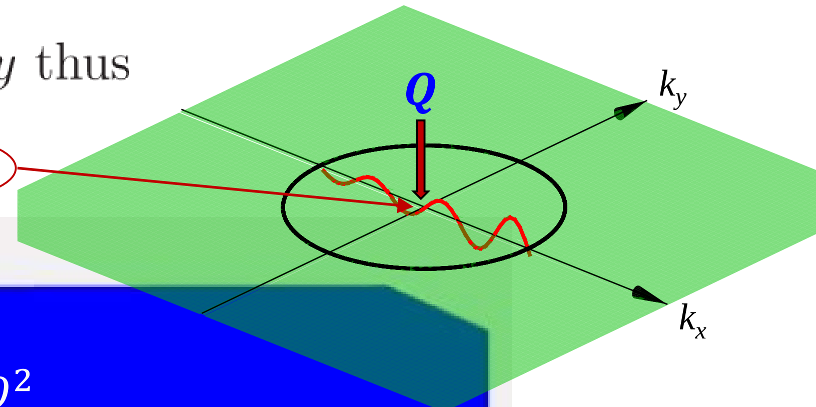


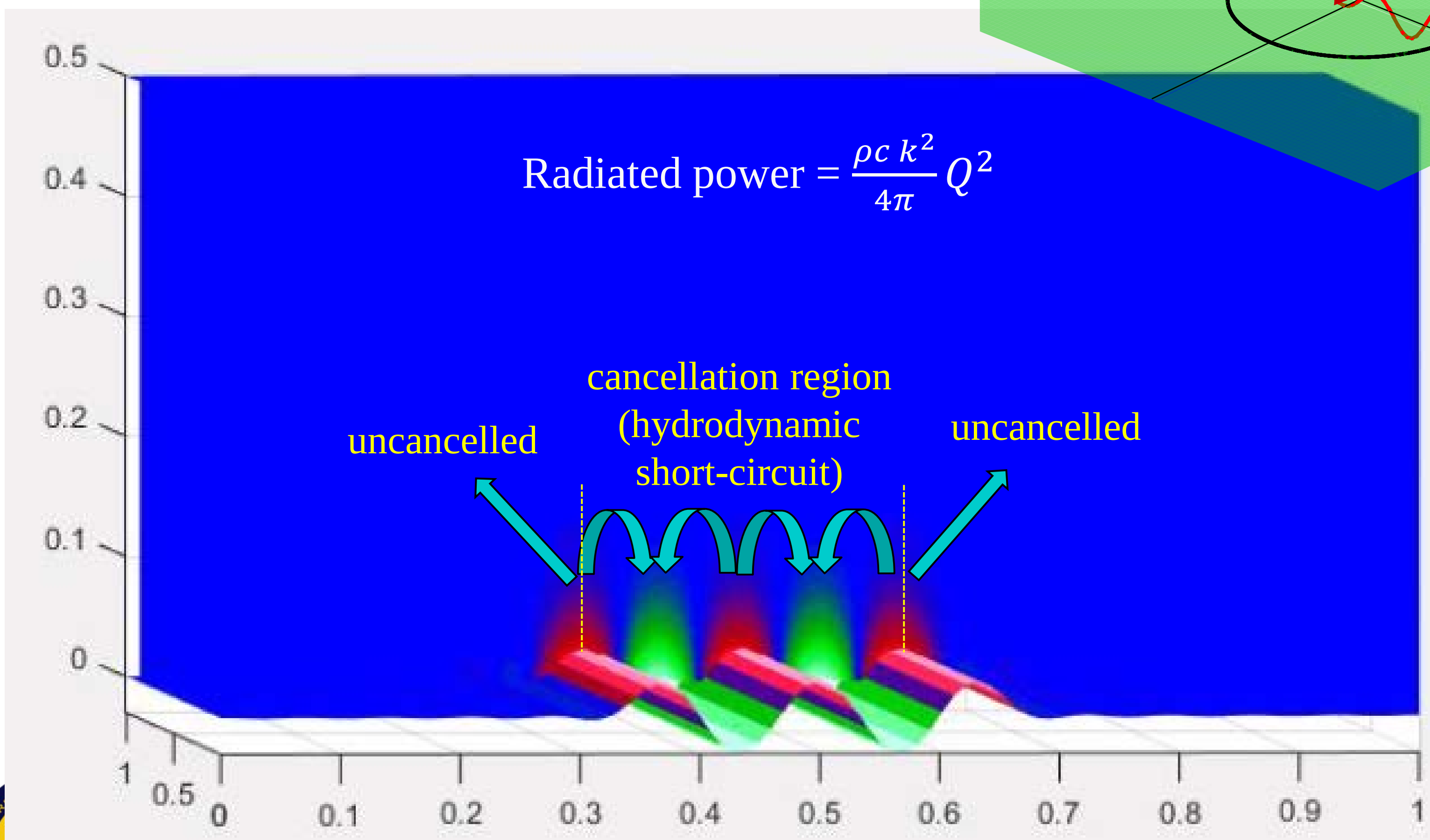
Edge Mode Radiation of Slow waves (kL<1, monopole term)

- The uncancelled volume velocity of slow waves ⬄ Value at the Origin

Volume velocity $Q \equiv \iint v_z(x,y,0)\,dx\,dy$ thus

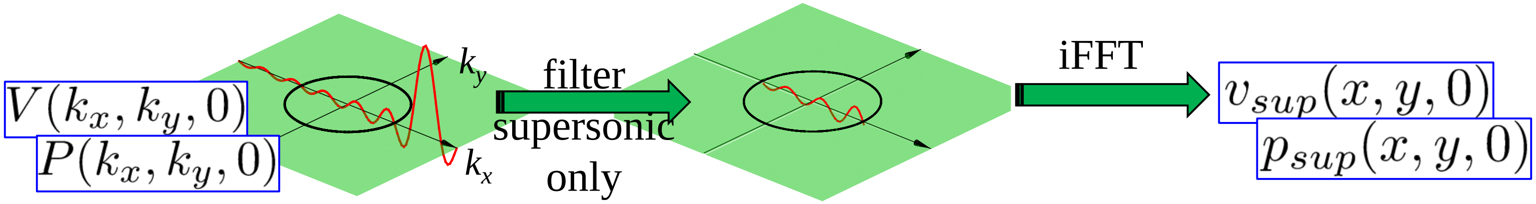
$$Q = \iint v_z(x,y,0)e^{ik_x x}e^{ik_y y}dxdy\Big|_{k_x=k_y=0}$$





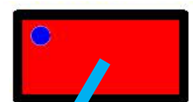
Supersonic Image of Surface INTENSITY

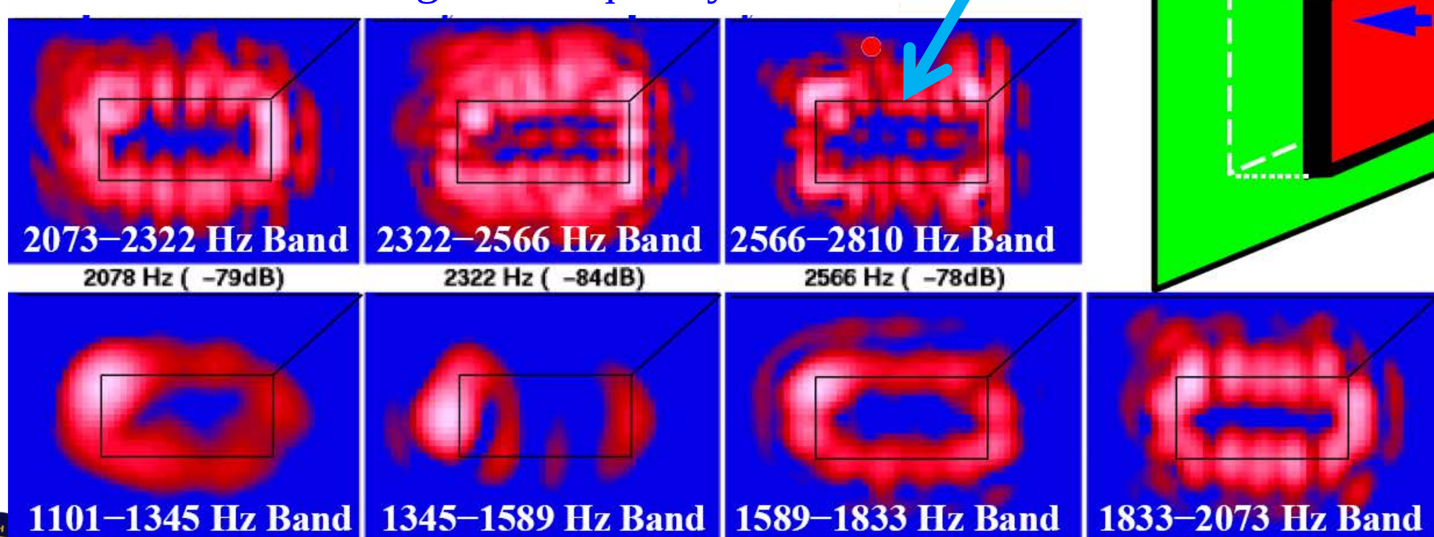
$$I_z(x,y,0) = \frac{1}{2}\Re[p_{sup}(x,y,0)v_{sup}(x,y,0)^*]$$

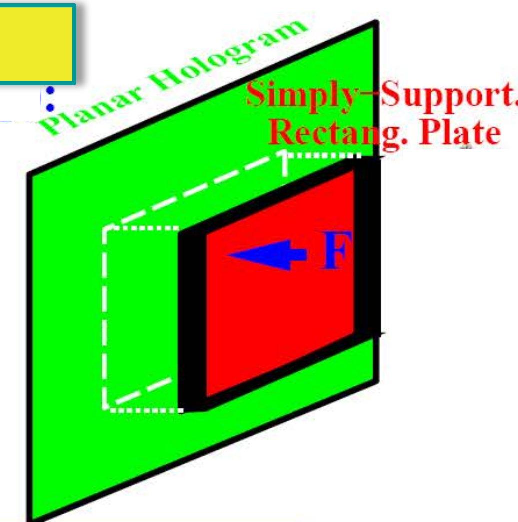


Example: a point driven thin plate (substrate)

Power/unit area image vs Frequency Band

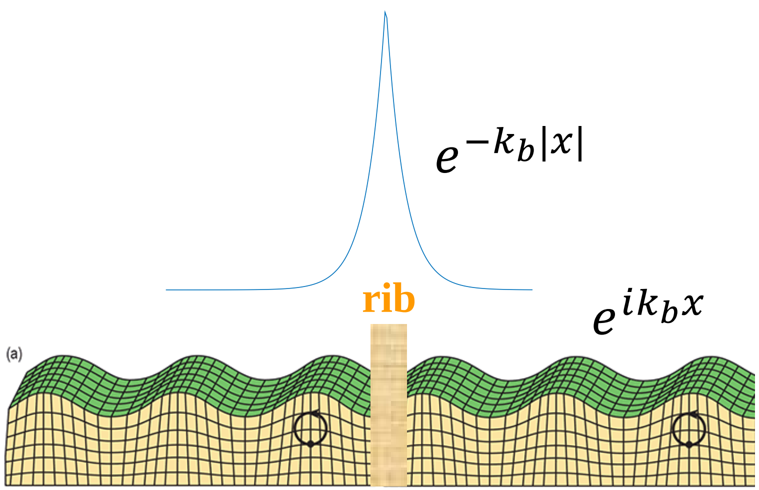


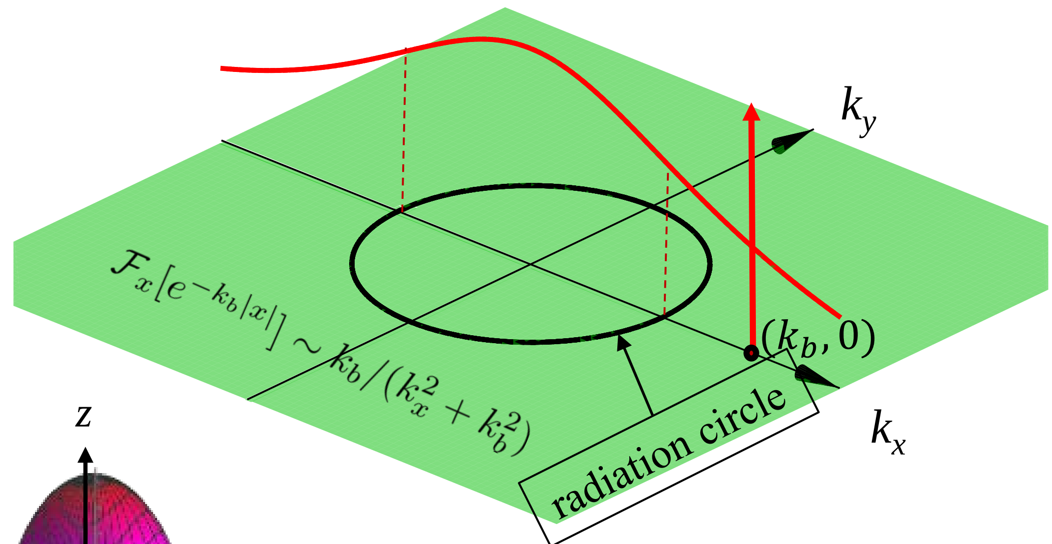


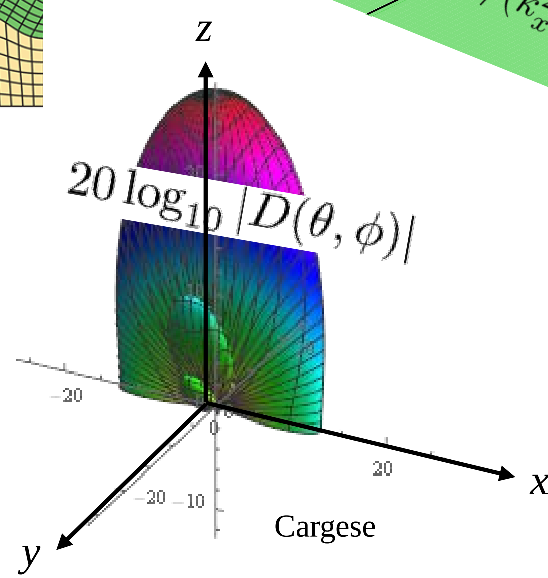


Radiation from Evanescent Near-fields

What is an evanescent near-field? Consider thin plate equation, solutions are Traveling: e^{ik_bx}, Evanescent: $e^{-k_b|x|}$ with $e^{-i\omega t}$. An evanescent near-field is created by oscillating loads, discontinuities (thickness change, density change, rib attachments, free ends, etc.)

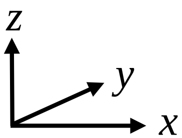






Radiation from Evanescent Near-fields ($e^{-k_b|x|}$)

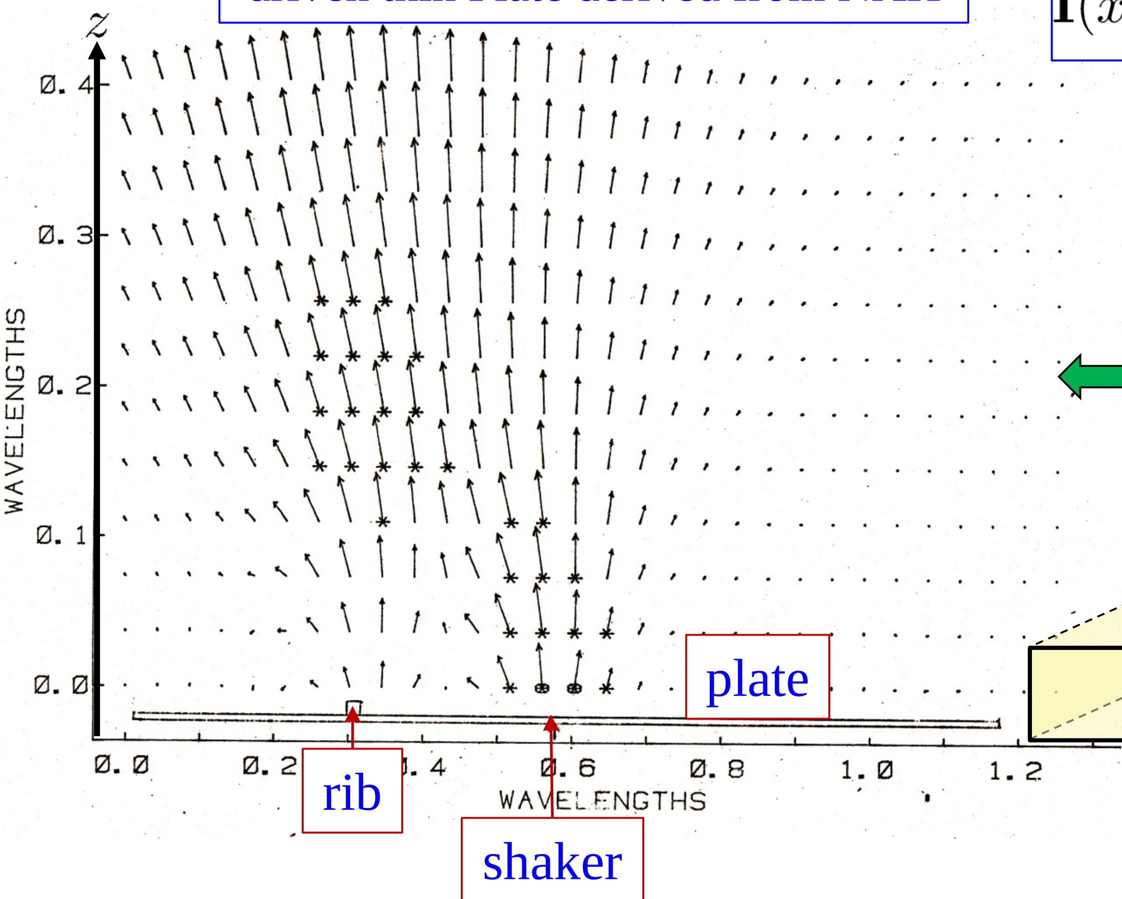
Example: Point driven plate with a rib attached

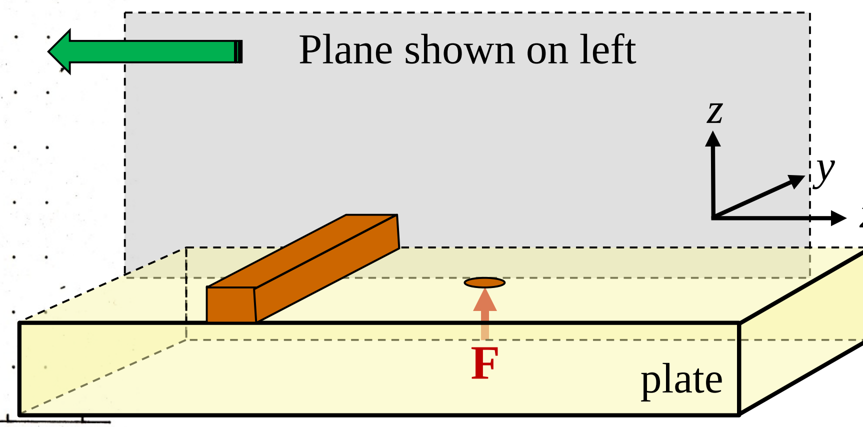


Flow of Intensity above a Point-driven thin Plate derived from NAH

$$\vec{\mathbf{I}}(x,y,z) = \frac{1}{2}\Re[p(x,y,z)\vec{\mathbf{v}}(x,y,z)^*]$$

$$\text{Power} = \int_S \vec{\mathbf{I}}(x,y,z_0)\cdot\hat{\mathbf{n}}\;dx\,dy$$





OUTLINE

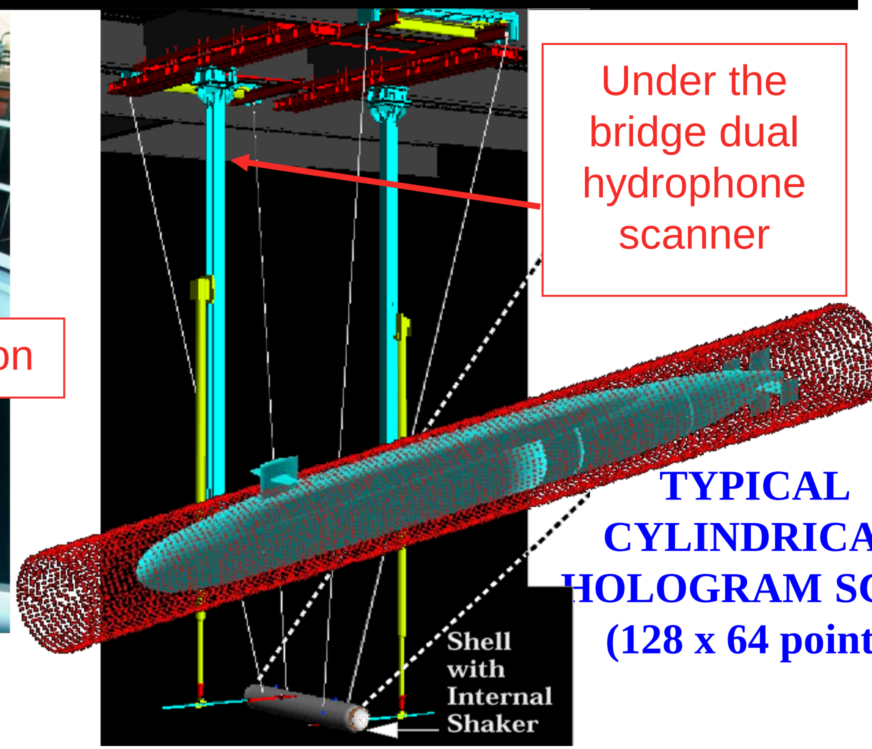
- Slowness space solutions to Helmholtz equations (also called *k*-space, angular spectrum)
- Coupling to the Fluid above
- Radiation to the Far-field
- Sources of radiation on interface (concept of super-sonic imaging?)
- Slowness surface examples from Experiments at NRL
  - Towards understanding the physics of wave propagation in the substrate - Beaming
- Dispersion Space – Bandgaps
  - Application to a metamaterial

Examples from Research at NRL – Near-field Holography

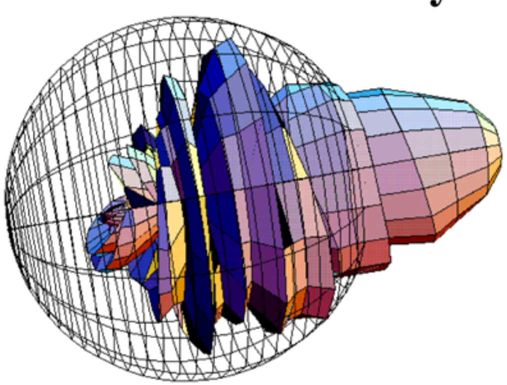
- SLOWNESS SPACE leading to global diagnostics of fluid-structure interactions to study Vibration, Radiation and Scattering (0-60kHz)

Pool Research Management: Dr. Brian Houston

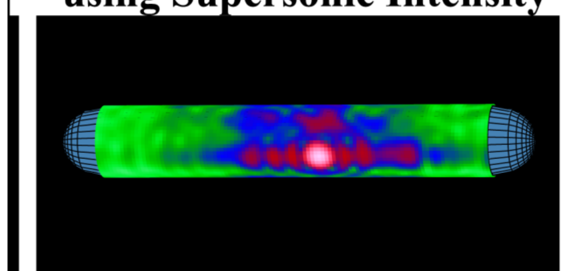
Naval Research Laboratory
10M$ Pool Facility

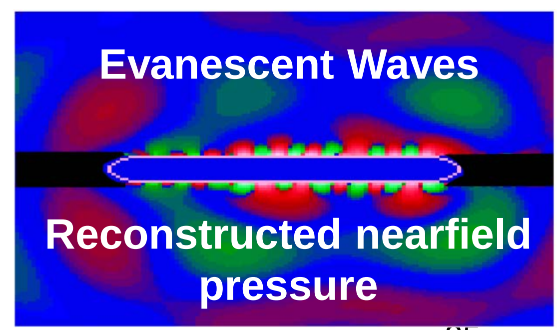


Farfield Directivity

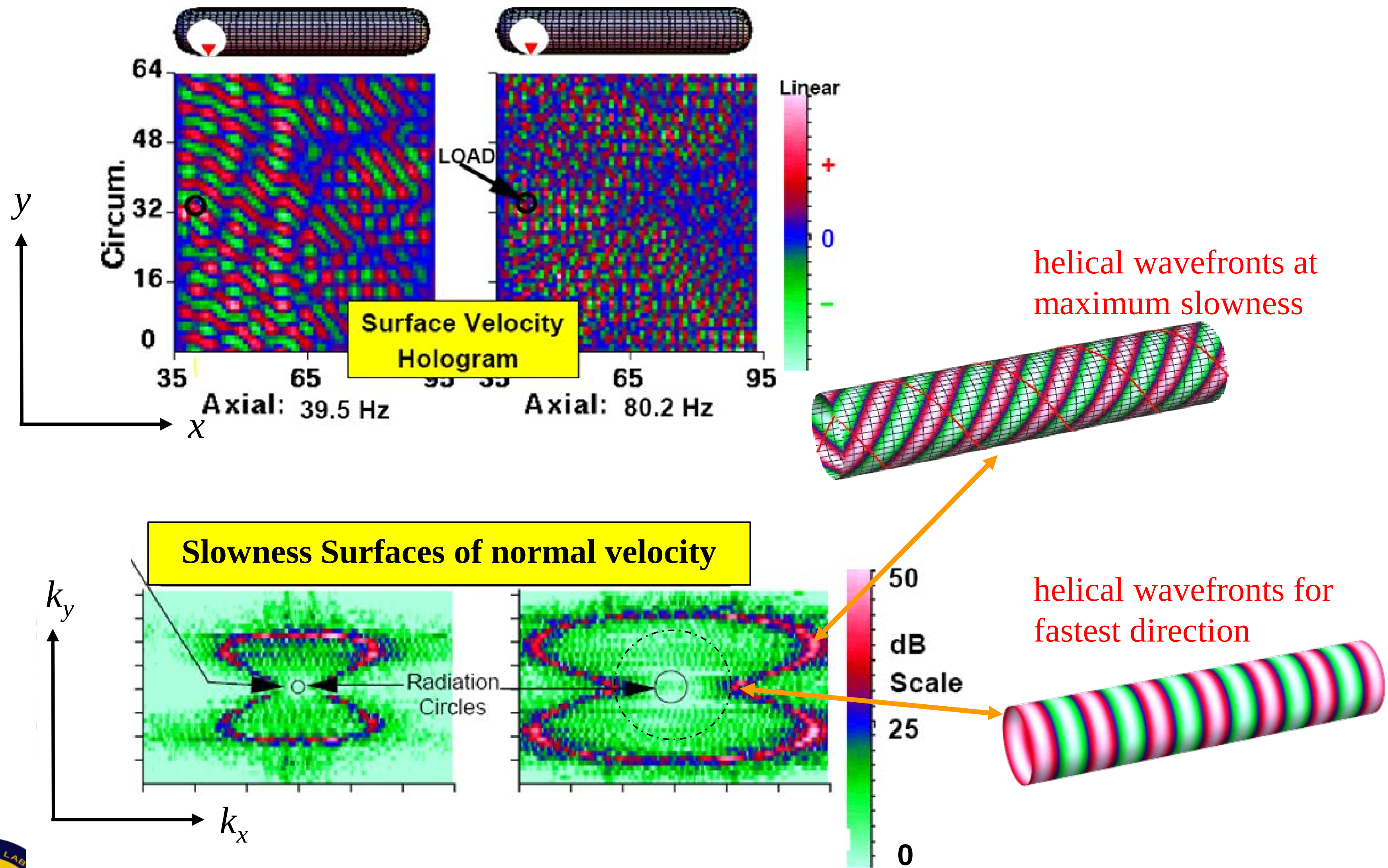


Localization of Hull 'Hot Spots' using Supersonic Intensity





Examples: Non-isotropic substrate – (Point driven Cylindrical Shell)



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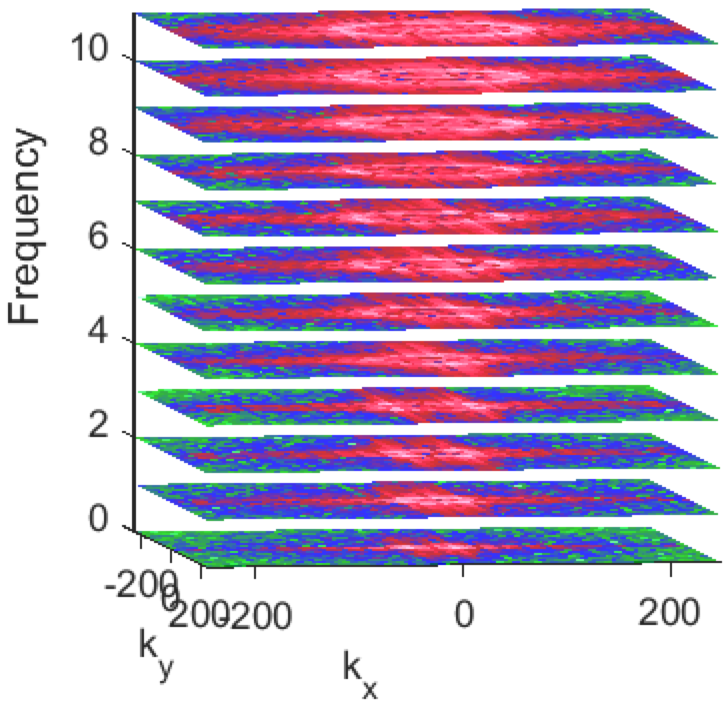
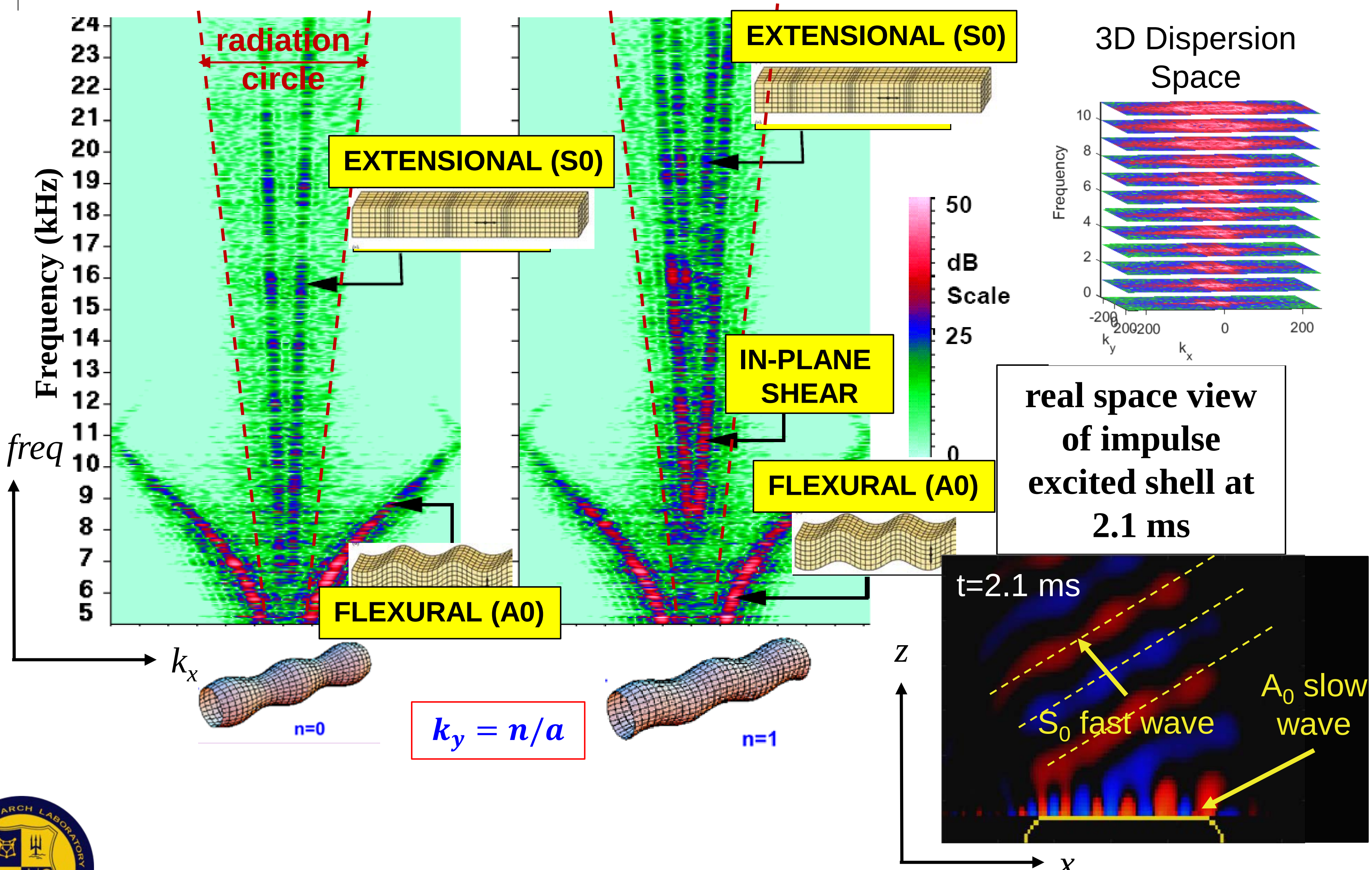
Dispersion Space

Stacking of slowness surfaces with frequency

Dispersion Space (k_x, k_y, f)

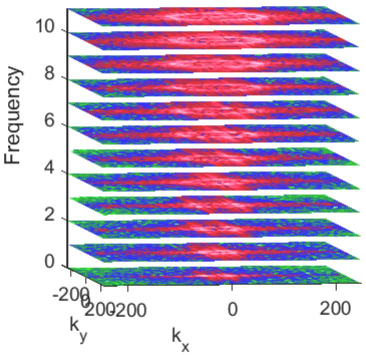


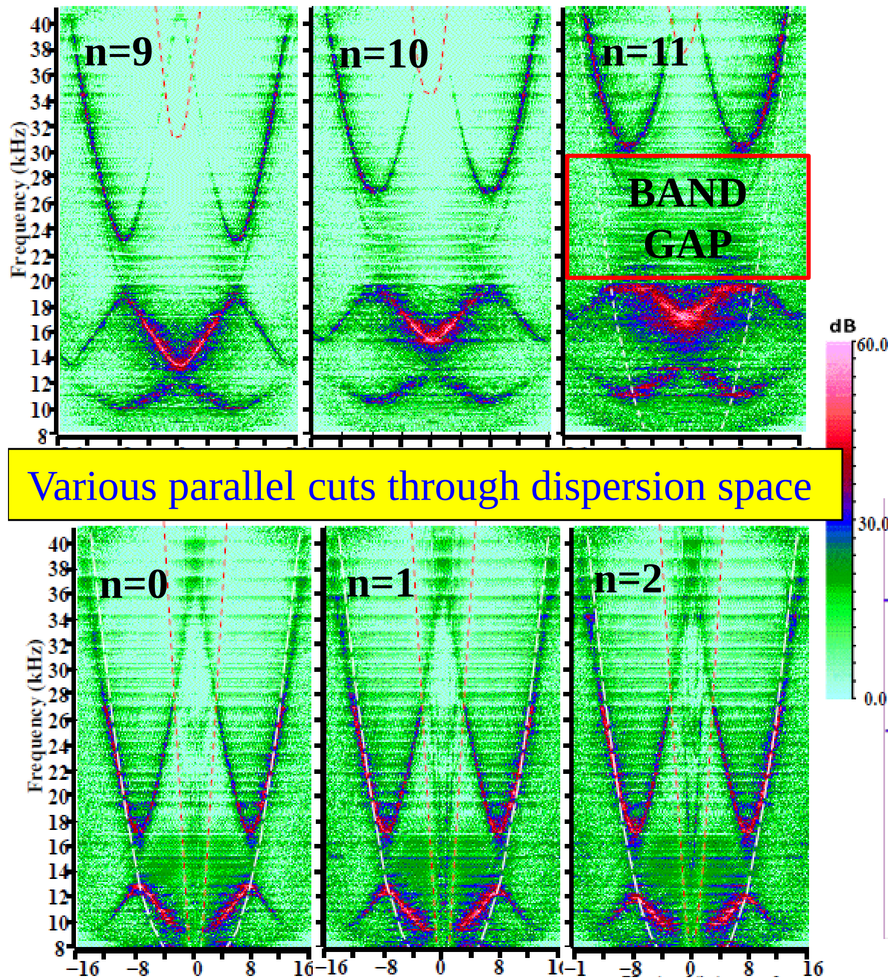
Dispersion Space – Example for a shell without ribs – Three types of waves that exist on this structure are identified & characterized

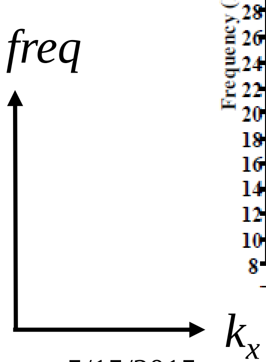


Effects of a periodic structure (ribs) – DIRECTIONAL BANDGAPS

3D Dispersion Space

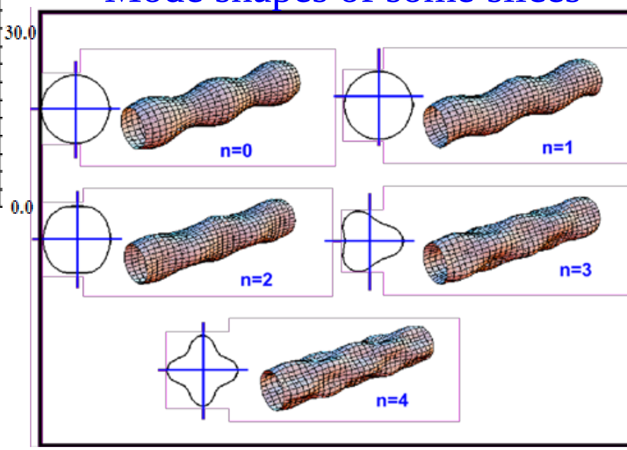




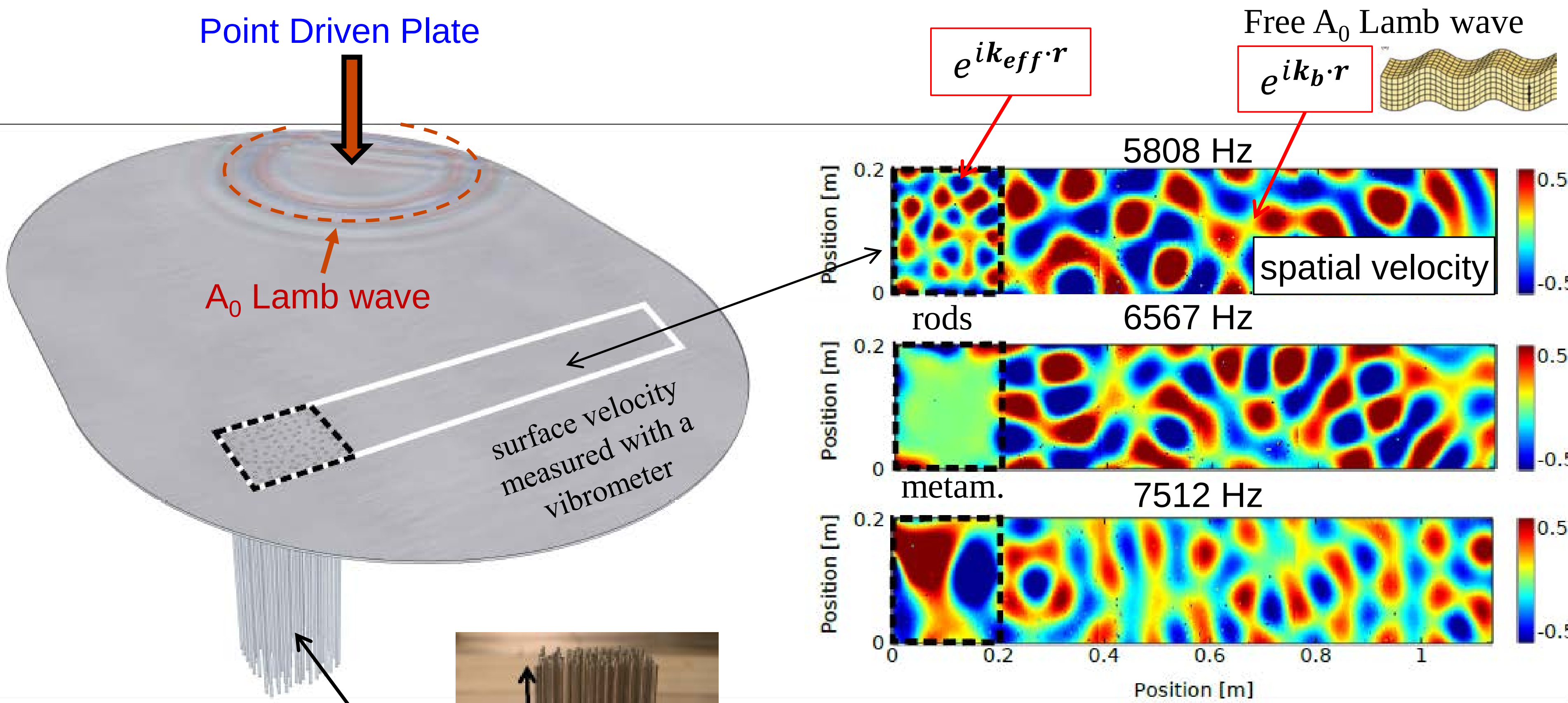


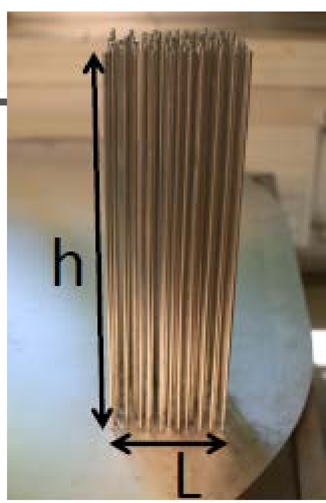
- Each parallel cut represents a mode shape of structure since $k_y = \frac{n}{a}$ (a is shell radius)

Mode shapes of some slices



Metamaterial Example*

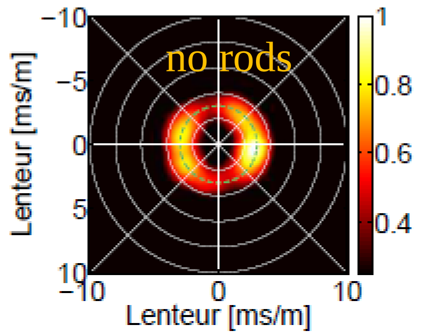




Metamaterial
10x10 lattice of long rods
glued to plate

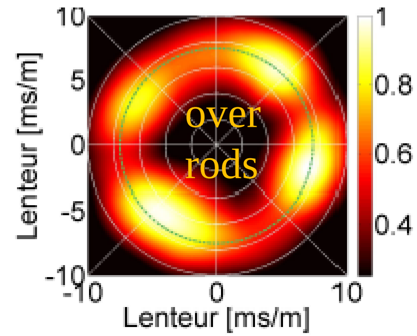
Experimental Demonstration of Ordered and Disordered Multiresonant Metamaterials for Lamb Waves, M. Rupin, F. Lemoult, G. Lerosey, P. Roux, PRL 112, 234301 (2014)

Metamaterial Dispersion



1.6-2.1 kHz

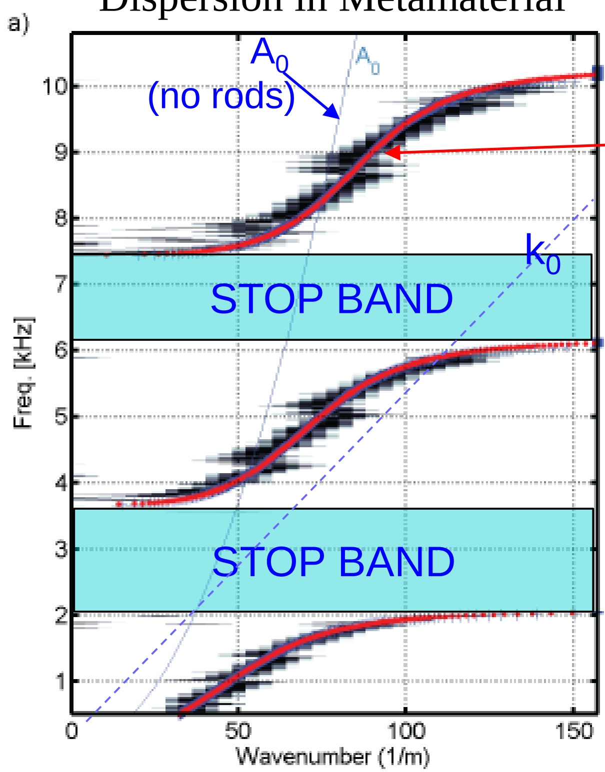
c = 333m/s

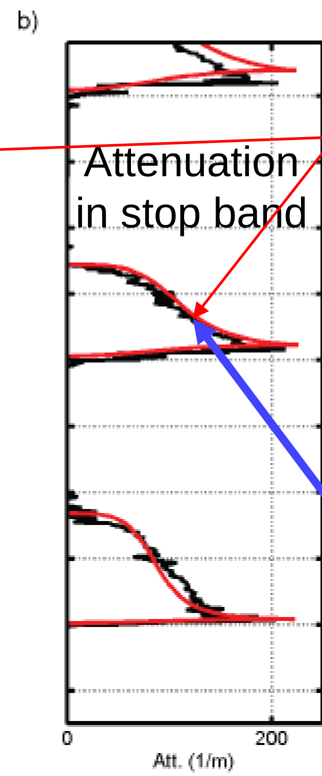


c = 133m/s

(b)

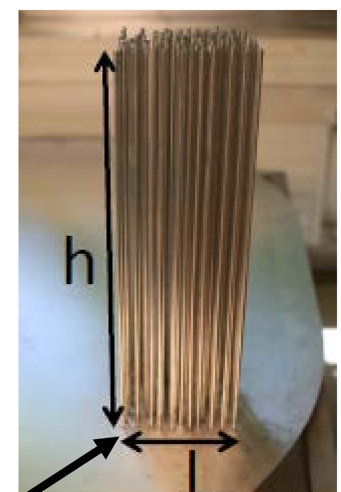
Dispersion in Metamaterial

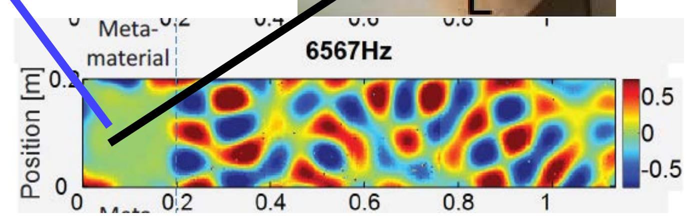




1D Theory*

Theory of multiresonant metamaterials for A_0 Lamb waves, Williams, Roux, Rupin, Kuperman, PRB **91**, 104307 (2015)





SUMMARY

- Slowness space – Monochromatic waves
    - Supersonic and subsonic waves
    - Radiation Circle
    - Far-field Radiation: Directivity pattern
    - Radiation from subsonic waves
    - Radiation from flexural near-fields
    - SuperSonic Imaging: Sources of Sound
    - Dispersion space: Band Gaps, Wave ID
    - Metamaterials