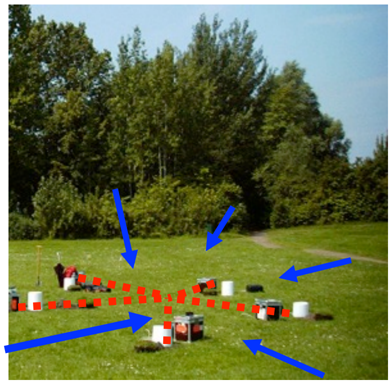
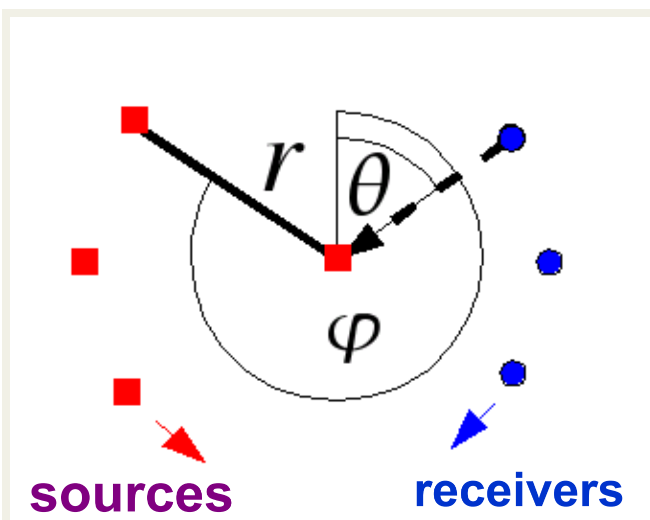
Spatial autocorrelation coefficient: Average spatial coefficient (2D circular array)

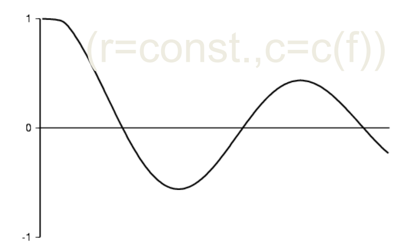




Aki (1957): azimuthal averaged spatial autocorrelation coefficients

$$\bar{\rho}(\mathbf{r},\omega_0) = \frac{1}{\pi}\int_0^{\pi} \rho(\mathbf{r},\varphi,\omega_0)\,\mathbf{d}(\theta-\varphi)$$

$$\boxed{\bar{\rho}(r,\omega_0) = J_0\left(\frac{\omega_0 r}{c(\omega_0)}\right)}$$

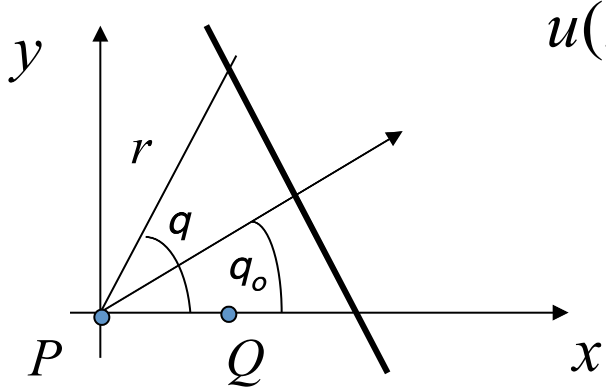


Evaluating k at different frequencies makes it possible to obtain the dispersion curve $C(\omega)$.

The method relies on the hypothesis of the stationnarity of the noise and requires specific array design to perform the azimuthal average.

Another approach consists of using only two points and to rely on long term average to produce the azimuthal average.

Let us consider a plane wave in 2D:



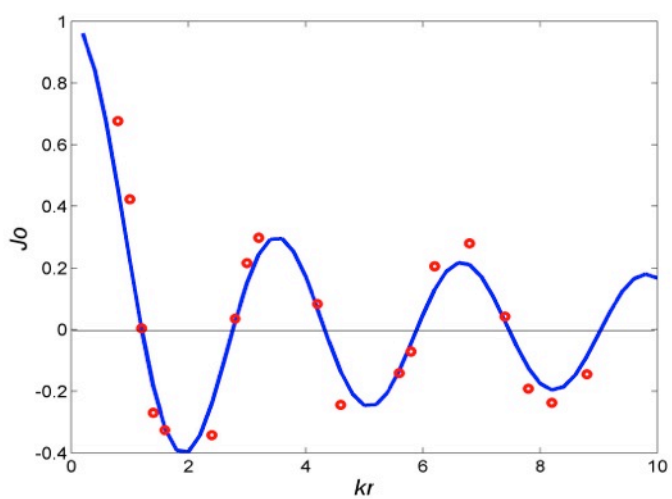
$$u(r,\theta,\omega) = F(\omega)\exp(-\mathrm{i}kr\cos(\theta-\theta_0))$$

$$\frac{u^P u^{Q^*}}{\left|u^P\right|\left|u^Q\right|} = e^{+ikr\cos\theta_0}$$

$$\boxed{\langle \rho(r,\omega)\rangle = \left\langle \frac{u^P u^{Q^*}}{|u^P||u^Q|} \right\rangle = \left\langle e^{ikr\cos\theta_0} \right\rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{ikr\cos\theta_0}\, d\theta_0 = J_0(kr)}$$

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{m=0}^{\infty}\varepsilon_m \mathrm{i}^m J_m(kr)\cos m\theta_0\right)d\theta_0$$

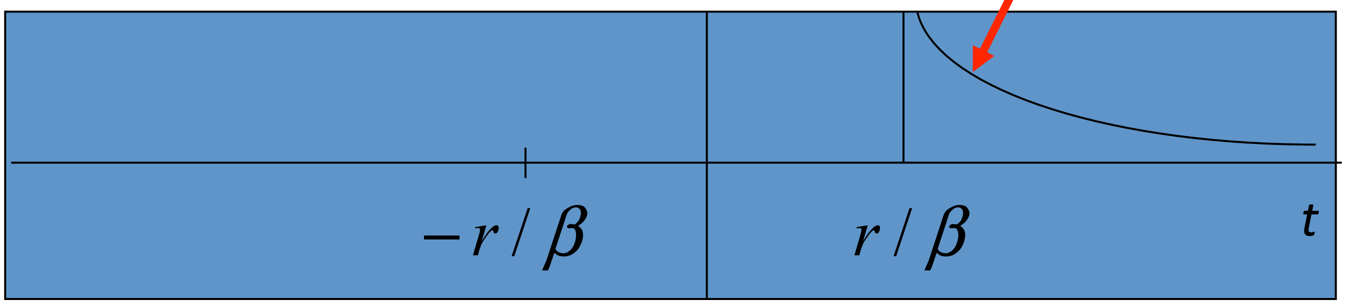
azimuthal average of the spatial cross-correlation



Causality

$$G = \frac{1}{4\mathrm{i}\mu} H_0^{(2)}\left(\frac{\omega r}{\beta}\right)$$

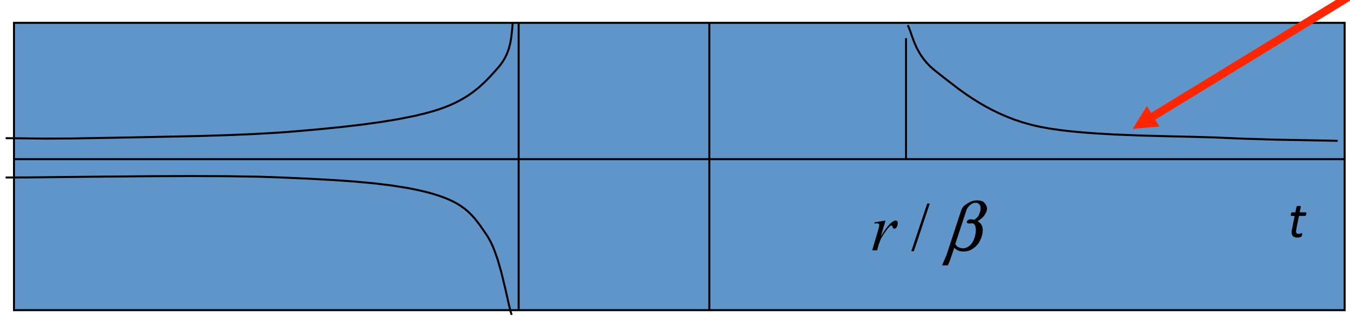
$$G = \frac{1}{2\pi\mu} \frac{H\left(t - {r}/{\beta}\right)}{\sqrt{t^2 - {r^2}/{\beta^2}}}$$



$G/2$

(Re)

(Im)



(Im, Re)

$$G_{22}(r,\omega) = \frac{1}{4\mu}\left\{ -Y_0\left(\frac{\omega r}{c}\right) - iJ_0\left(\frac{\omega r}{c}\right)\right\}$$

$$J_0\left(\frac{\omega r}{c(\omega)}\right) = -4\mu \,\mathrm{Im}\left(G_{22}(r,\omega)\right) \qquad r = \left|P,Q\right|$$

$$\boxed{\boxed{\mathrm{Im}\left(G_{22}^{PQ}\right) = \frac{-1}{4\mu}\left\langle \frac{u_2(P)u_2^*(Q)}{\left|u_2(P)\right|\left|u_2(Q)\right|}\right\rangle}}$$

P-SV case

Green function in 2D

$$G_{ij} = \frac{i}{4\rho\omega^2}\left\{-\delta_{ij}k^2H_0^{(2)}(kr) + \frac{\partial^2}{\partial x_i \partial x_l}\left[H_0^{(2)}(qr) - H_0^{(2)}(kr)\right]\delta_{lj}\right\}$$

$$G_{ij}(P,Q) = \frac{-\mathrm{i}}{8\rho}\left\{A\delta_{ij} - B\left(2\gamma_i\gamma_j - \delta_{ij}\right)\right\} \qquad \gamma_j = \frac{x_j - \xi_j}{r}$$

$$A = \frac{H_0^{(2)}(qr)}{\alpha^2} + \frac{H_0^{(2)}(kr)}{\beta^2} \qquad B = \frac{H_2^{(2)}(qr)}{\alpha^2} - \frac{H_2^{(2)}(kr)}{\beta^2}$$

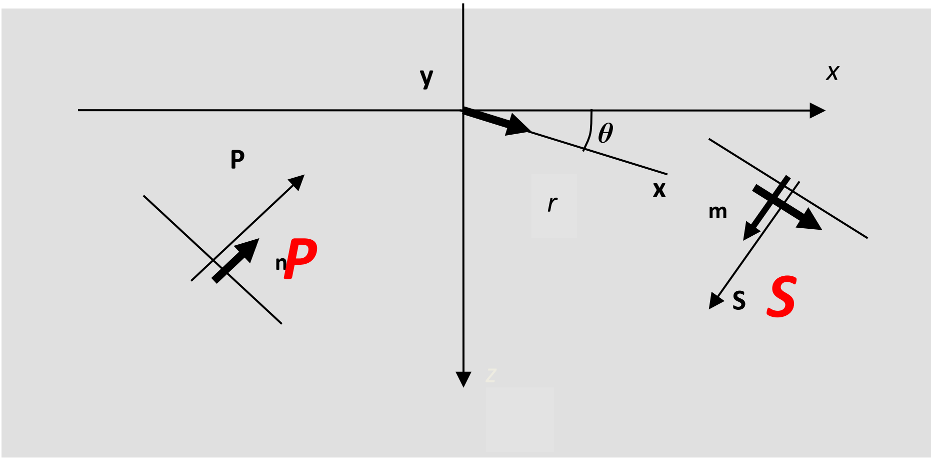
$$\alpha = \sqrt{\frac{\lambda + 2\mu}{\rho}} \qquad \beta = \sqrt{\frac{\mu}{\rho}} \qquad r = |P,Q|$$

THE 2D VECTOR CASE

$$\beta^2 \frac{\partial^2 u_i}{\partial x_j \partial x_j} + (\alpha^2 - \beta^2)\frac{\partial^2 u_j}{\partial x_i \partial x_j} = \frac{\partial^2 u_i}{\partial t^2}$$

Summation of P and S plane waves:

$$u_l(\mathbf{x},\omega,t) = P(\omega,\phi) n_l \exp(-\mathrm{i}\frac{\omega}{\alpha} x_j n_j) + S(\omega,\psi) m_l \exp(-\mathrm{i}\frac{\omega}{\beta} x_j m_j)$$



Correlation:

$$u_l(\mathbf{y}) u_s^*(\mathbf{x}) =$$

$$(P^2 n_l n_s + SP^* m_l n_s)\exp(\mathbf{i}kr\cos[\phi-\theta]) +$$

$$(S^2 m_l m_s + PS^* n_l m_s)\exp(\mathbf{i}kr\cos[\psi-\theta])$$

Azimuthal average:

$$\langle \bullet \rangle = \frac{1}{4\pi^2} \int_0^{2\pi} d\phi \int_0^{2\pi} \bullet \, d\psi$$

$$P^2\alpha^2 = \varepsilon S^2\beta^2$$

Equipartition (ε=1):

$$E_S / E_P = \left(\frac{\alpha}{\beta}\right)^2 \frac{1}{\varepsilon}$$

$$\langle u_i(\mathbf{y}) u_j^*(\mathbf{x}) \rangle = \frac{S^2\beta^2}{2} \left\{ A\delta_{ij} - B(2\gamma_i\gamma_j - \delta_{ij}) \right\}$$

$$A = \varepsilon \frac{J_0(qr)}{\alpha^2} + \frac{J_0(kr)}{\beta^2} \text{ and } B = \varepsilon \frac{J_2(qr)}{\alpha^2} - \frac{J_2(kr)}{\beta^2}$$

And finally if ε=1

$$\langle u_i(\mathbf{y}, \omega) u_j^*(\mathbf{x}, \omega) \rangle \equiv -8 E_S k^{-2} \operatorname{Im}\left[G_{ij}(\mathbf{x}, \mathbf{y}, \omega) \right]$$

Formally, same result in 3D (Sánchez-Sesma and Campillo, BSSA 2006)

Arbitrary medium: an integral representation written in the frequency domain

(see e.g. Weaver et al. 2004, or Snieder, 2007)

$$G_{12} - G_{12}^{*} = \boxed{\frac{4i\omega\kappa}{c}\int_{\mathcal{V}} G_{1x}G_{2x}^{*}\,dV} + \boxed{\oint_{S}\left[G_{1x}\vec{\nabla}\left(G_{2x}^{*}\right) - \vec{\nabla}\left(G_{1x}\right)G_{2x}^{*}\right]\overrightarrow{dS}}$$

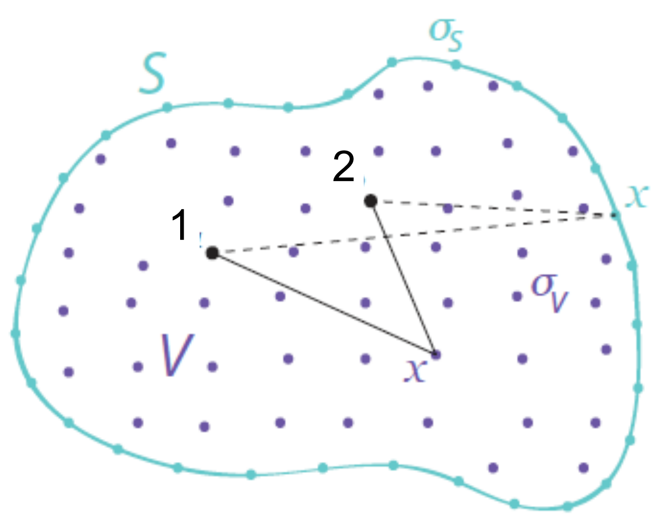
Volume term

Surface term

FT of $G(-t)$

Absorption coefficient

FT of $G(t)$



Helmholtz equation $G_{1x} = G(\vec{r}_1, \vec{x}; \omega)$

$$\Delta G_{1x} + V(\vec{x}) G_{1x} + (k + i\kappa)^2 G_{1x} = \delta(\vec{x} - \vec{r}_1)$$

where the potential $V(\vec{x})$ describes the scattering contribution does not extend to infinity.

As for the classical representation theorem, we consider a combination of the fields from source at 1 and 2 and compute the flux:

$$I = \oint_S \left[G_{1x} \vec{\nabla}\left(G_{2x}^*\right) - \vec{\nabla}\left(G_{1x}\right) G_{2x}^* \right] \overrightarrow{dS}$$

With the divergence theorem:

$$I = \int_{\mathcal{V}} \vec{\nabla} \left[G_{1x} \vec{\nabla}\left(G_{2x}^*\right) - \vec{\nabla}\left(G_{1x}\right) G_{2x}^* \right] dV$$

$$I=\int_{\mathcal{V}}\vec{\nabla}\left[G_{1x}\vec{\nabla}\left(G_{2x}^{*}\right)-\vec{\nabla}\left(G_{1x}\right)G_{2x}^{*}\right]dV$$

reduces to

$$I=\int_{\mathcal{V}}\left(G_{1x}\Delta G_{2x}^{*}-\Delta G_{1x}G_{2x}^{*}\right)dV$$

Using the definition of the GF:

$$\Delta G_{1x}=\delta\left(\vec{x}-\vec{r}_{1}\right)-V\left(\vec{x}\right)G_{1x}-\left(k+i\kappa\right)^{2}G_{1x}$$

we obtain:

$$I=G_{12}-G_{21}^{*}-\frac{4i\omega\kappa}{c}\int_{\mathcal{V}}G_{1x}G_{2x}^{*}\,dV$$

and finally:

$$G_{12}-G_{12}^{*}=\frac{4i\omega\kappa}{c}\int_{\mathcal{V}}G_{1x}G_{2x}^{*}\,dV+\oint_{S}\left[G_{1x}\vec{\nabla}\left(G_{2x}^{*}\right)-\vec{\nabla}\left(G_{1x}\right)G_{2x}^{*}\right]\overrightarrow{dS}$$

Surface term: $$G_{12} - G_{12}^{*} = \oint_S \left[G_{1x} \vec{\nabla}\left(G_{2x}^{*}\right) - \vec{\nabla}\left(G_{1x}\right) G_{2x}^{*} \right] \overrightarrow{dS}$$

κ =0 (no attenuation)

No source in the bulk

Surface term:

$$G_{12} - G_{12}^{*} = \oint_S \left[G_{1x} \vec{\nabla}\left(G_{2x}^{*}\right) - \vec{\nabla}\left(G_{1x}\right) G_{2x}^{*} \right] \overrightarrow{dS}$$

If the surface is taken in the far field of the medium heterogeneities

$$G_{1x} \sim \frac{1}{4\pi\left|\vec{x}-\vec{r}_1\right|} \exp\left(-ik\left|\vec{x}-\vec{r}_1\right|\right) \text{ and } \vec{\nabla}\left(G_{1x}\right) \sim i\vec{k}\, G_{1x}$$

and we obtain another widely used integral relation:

$$G_{12} - G_{12}^{*} = -2i\frac{\omega}{c} \oint_S G_{1x} G_{2x}^{*}\, dS$$

Talk by A. Curtis on integral representations and applications

Volume term: $$G_{12} - G_{12}^{*} = \frac{4i\omega\kappa}{c}\int_{\mathcal{V}} G_{1x}G_{2x}^{*}\,dV$$

κ is finite (attenuation)

S is assumed to be sufficiently far away, for its contribution to be neglected (spreading and attenuation)

An homogeneous infinite body with an even random distribution of sources

Green function
$$G(\vec{r}_1,\vec{r}_2;t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega \frac{1}{4\pi|\vec{r}_2-\vec{r}_1|}\exp\left[i\omega\left(t-\frac{|\vec{r}_2-\vec{r}_1|}{c}\right)\right]\exp\left(-\kappa|\vec{r}_2-\vec{r}_1|\right)$$

Contribution from
Random sources
$$P(\vec{r}_1;t)=\int_{-\infty}^{\infty}\int_{-\infty}^{t} d\vec{x}\,dt_x S(\vec{x},t_x)G(\vec{r}_1,\vec{x};t-t_x)$$

$$\langle S(\vec{x},t_x)S(\vec{x}',t_{x'})\rangle = Q^2\delta(t_x-t_{x'})\delta(\vec{x}-\vec{x}')$$

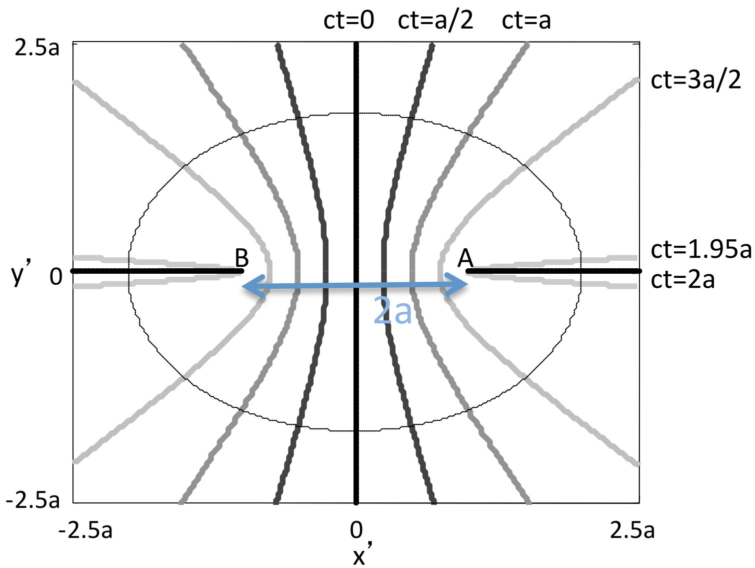
Correlation with finite duration
$$C(\vec{r}_1,\vec{r}_2;t)=C_{1,2}(t)=\frac{1}{T}\int_0^T P(\vec{r}_1;\tau)P(\vec{r}_2;t+\tau)\,d\tau$$

$$\frac{d}{dt}\langle C_{1,2}(t)\rangle = Q^2N\frac{c}{2\kappa}\left[G(\vec{r}_1,\vec{r}_2;t)-G(\vec{r}_1,\vec{r}_2;-t)\right]$$ with Q^2N being the noise power during T

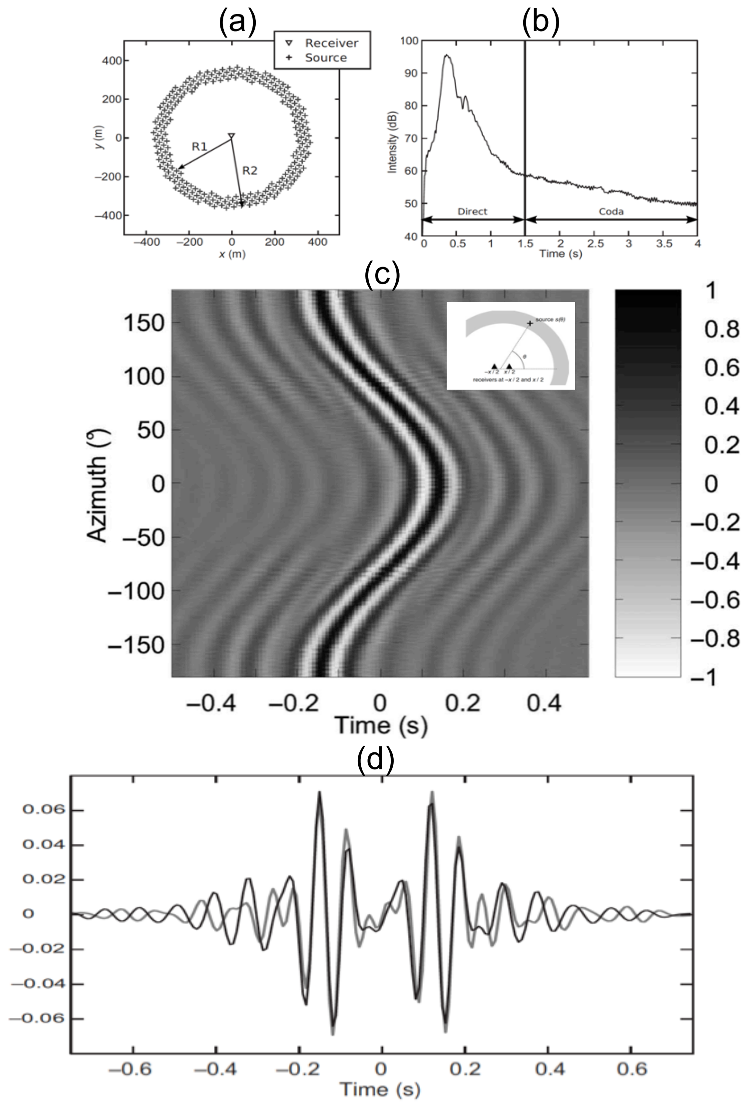
<time derivative of correlation> ≈ causal+acausal Green functions

Location of the sources that contribute to the correlation: the end fire lobes

Difference of travel time between A and B
wrt the position of the source

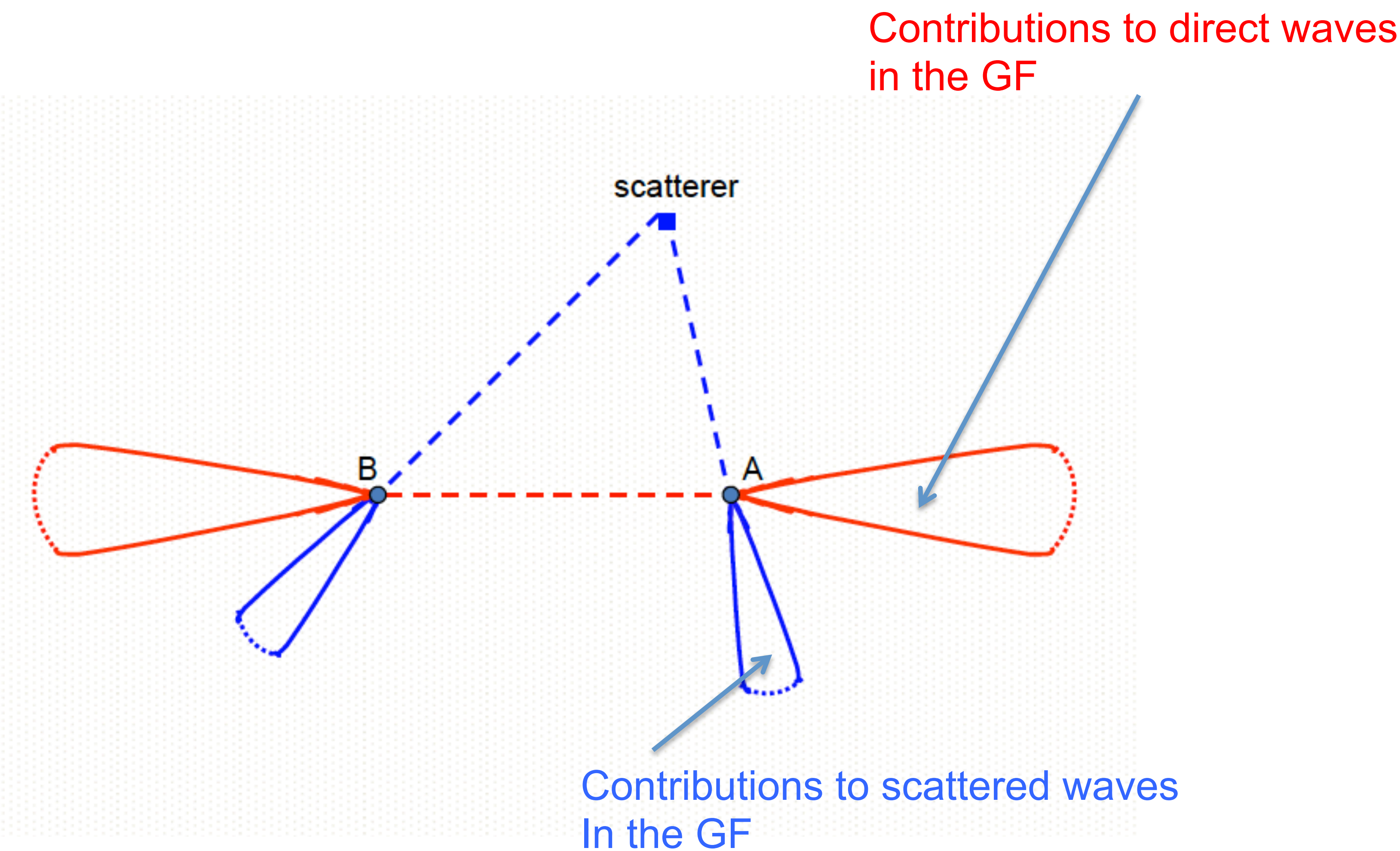


Stationary phase and end fire lobes



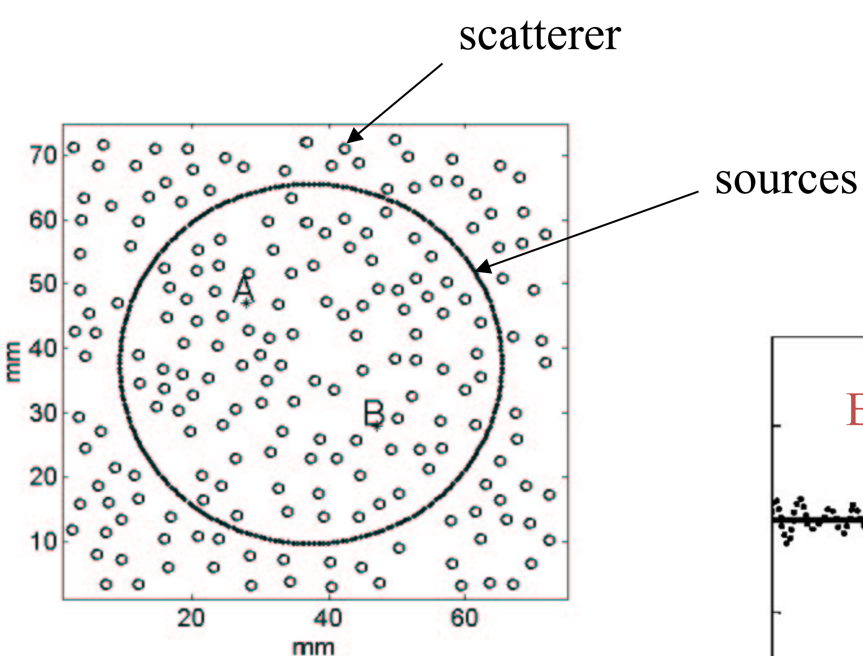
From Gouédard et al., 200X

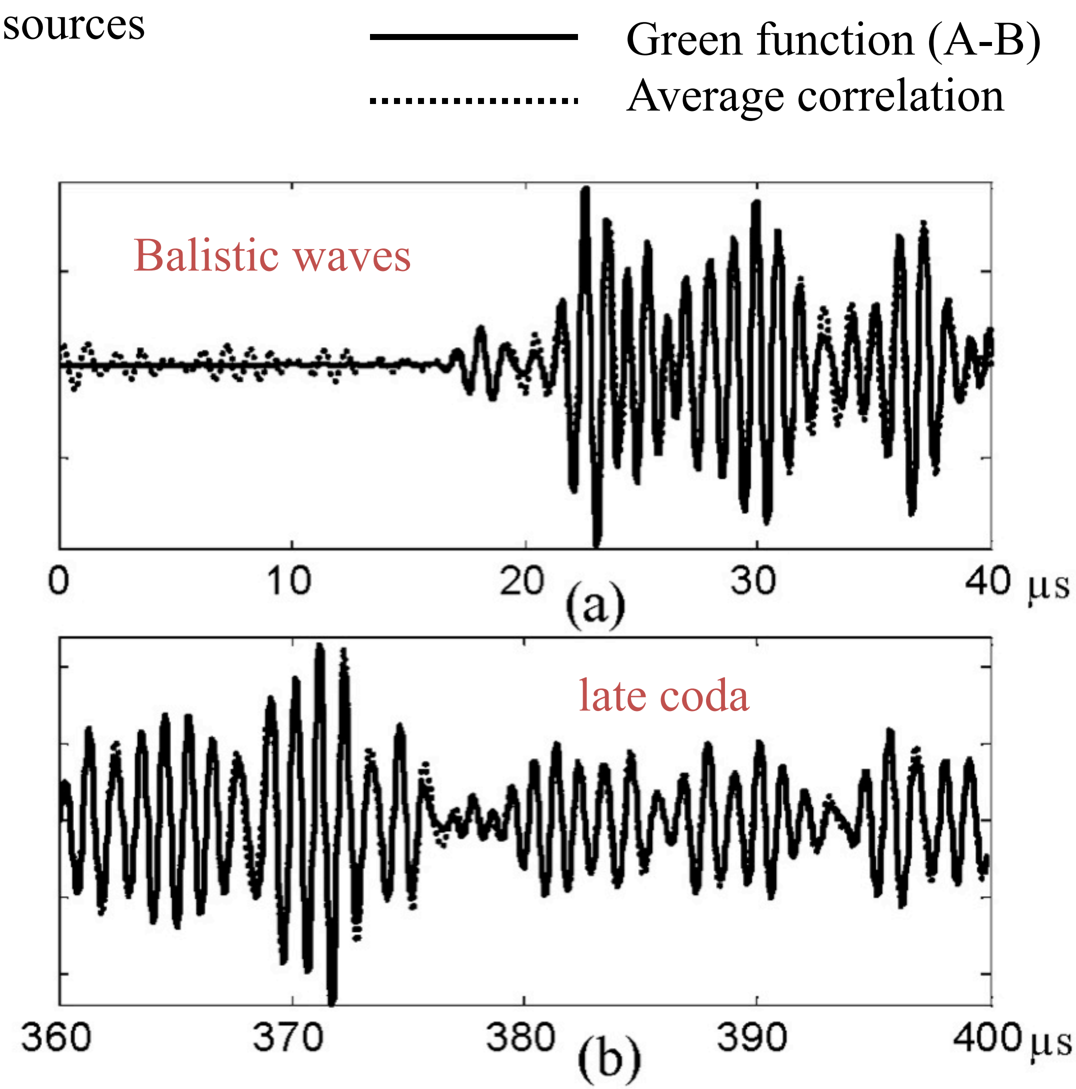
End fire lobes



Extension to scattered waves

A numerical experiment with an open medium (absorbing boundaries):

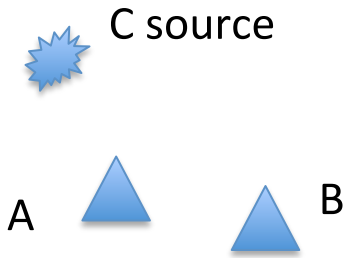




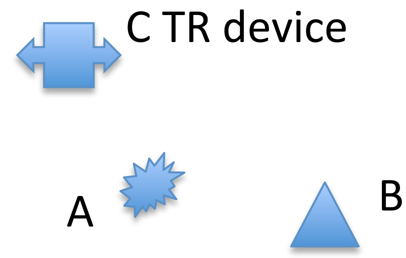
Physical interpretations

Time reversal

MOVIE : revers_water



Correlation and Time reversal
Focusing/virtual source in A



Equivalence in a reciprocal medium

- C source
- A et B receivers

- Correlation :

$S_{CA}(t) \times S_{CB}(t)$ =

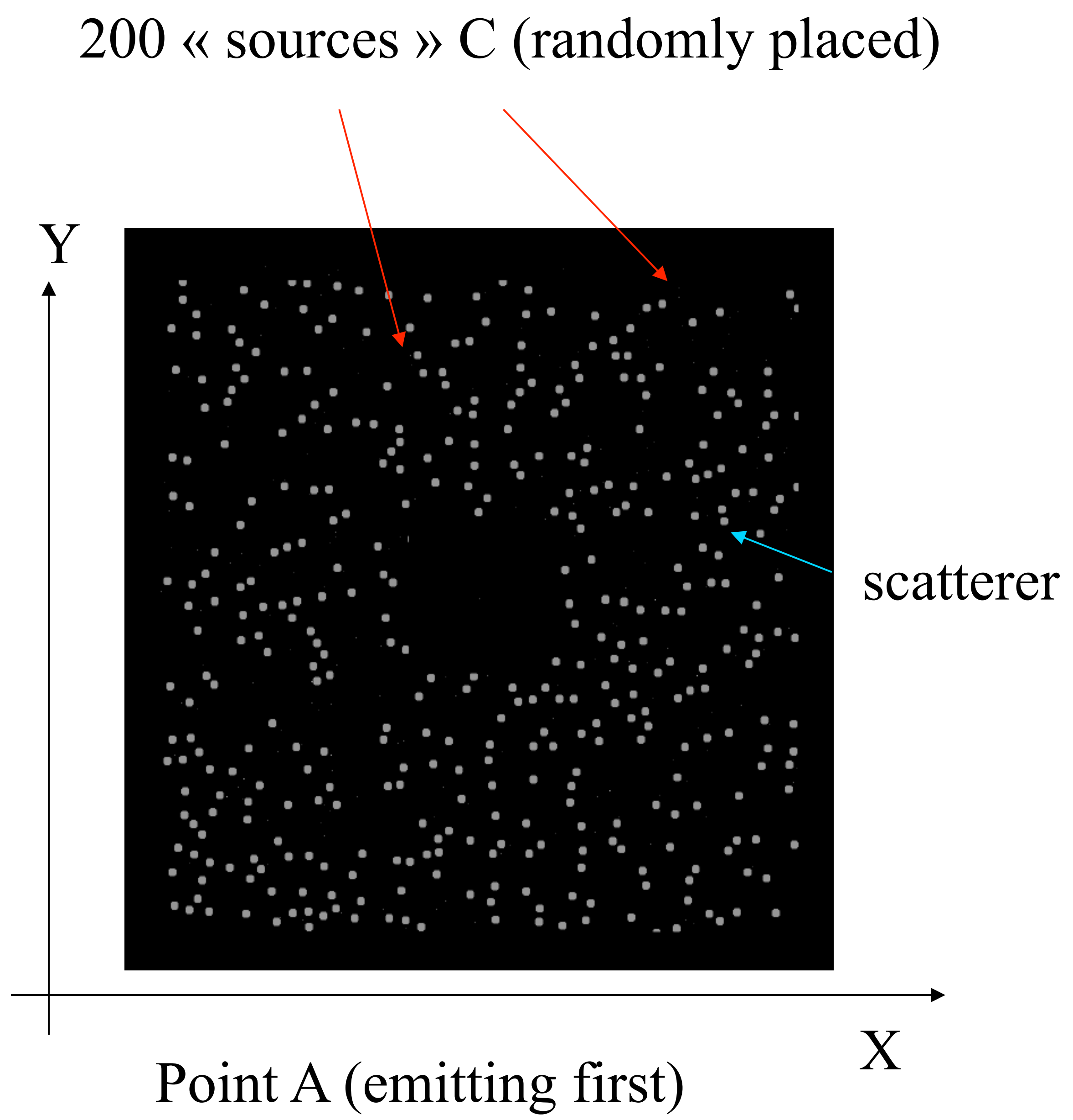
- A source
- C receiver ($S_{AC} = S_{CA}$)
- C emits the time reversed signal
- B receiver

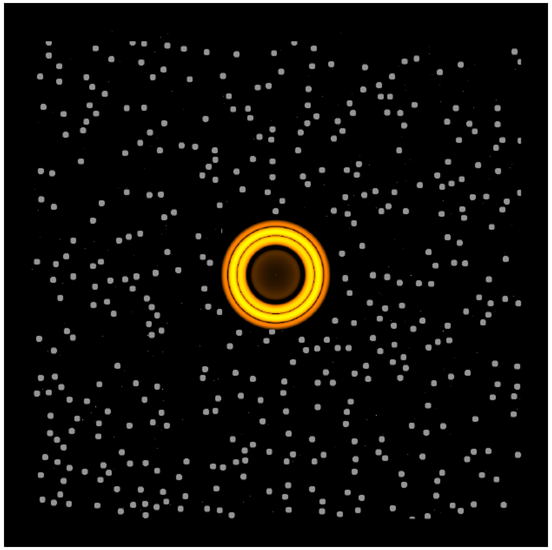
- Convolution :

$S_{CA}(t) \otimes S_{CB}(-t)$

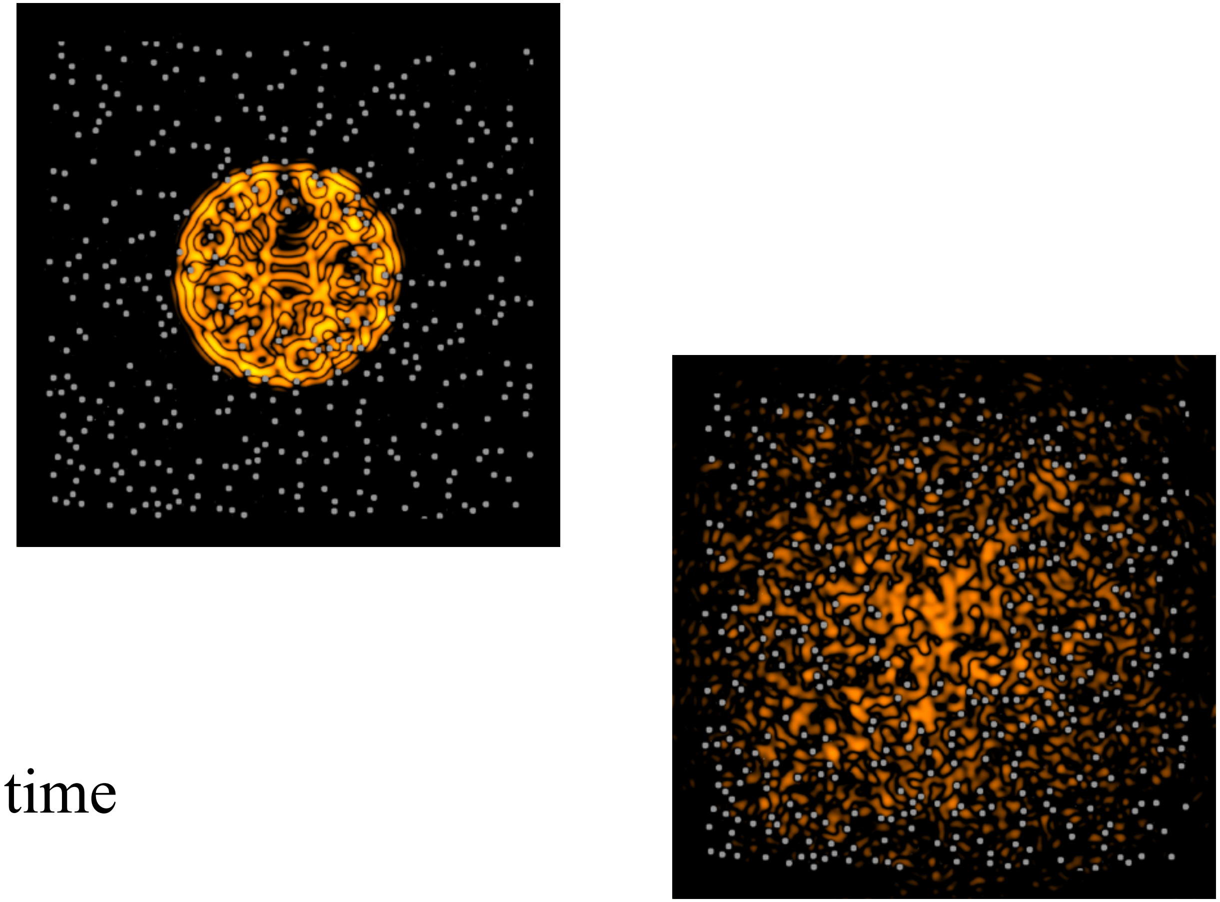
Derode et al., 2003++++

Numerical 2D FD simulation

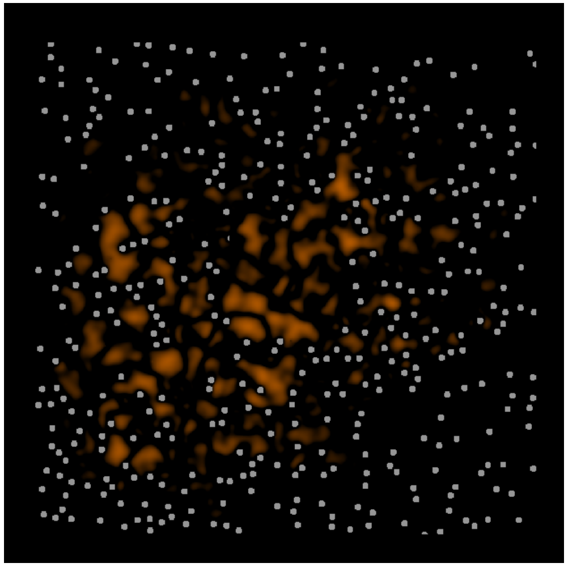




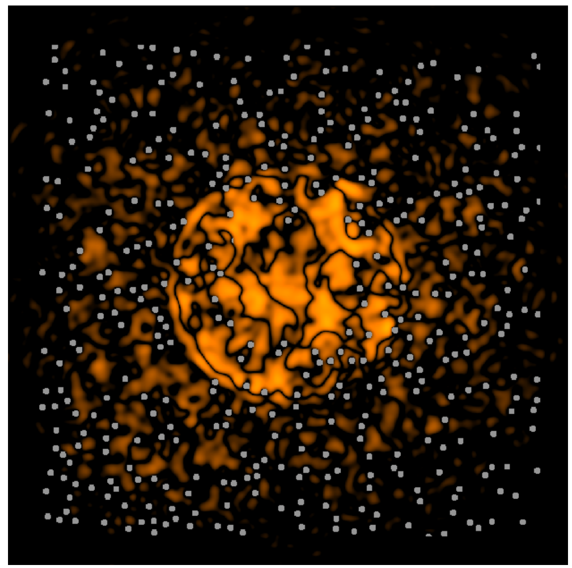
A pulse is emited in A
and recorded at point randomly
distributed



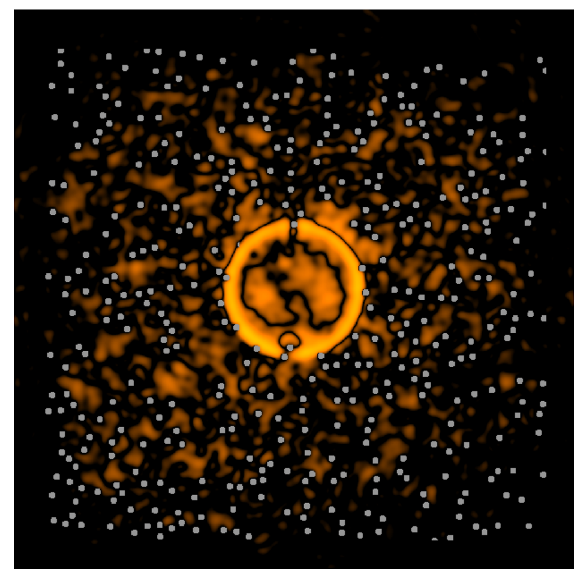
time



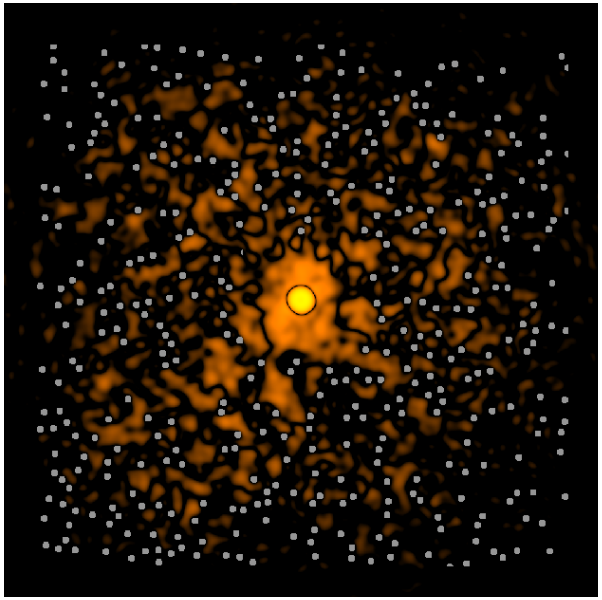
Re-emission from the points ‘C’
of the time-reversed signals
(map of cross-correlations)



Constructive
interferences of time-
reversed field

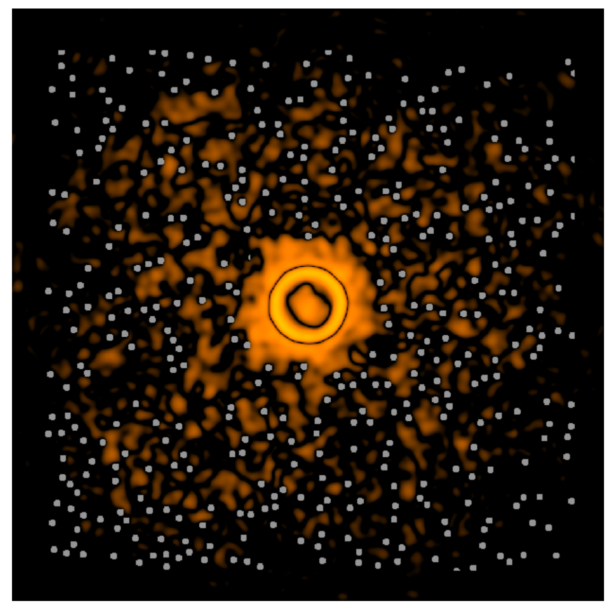


Converging field
: G(-t)

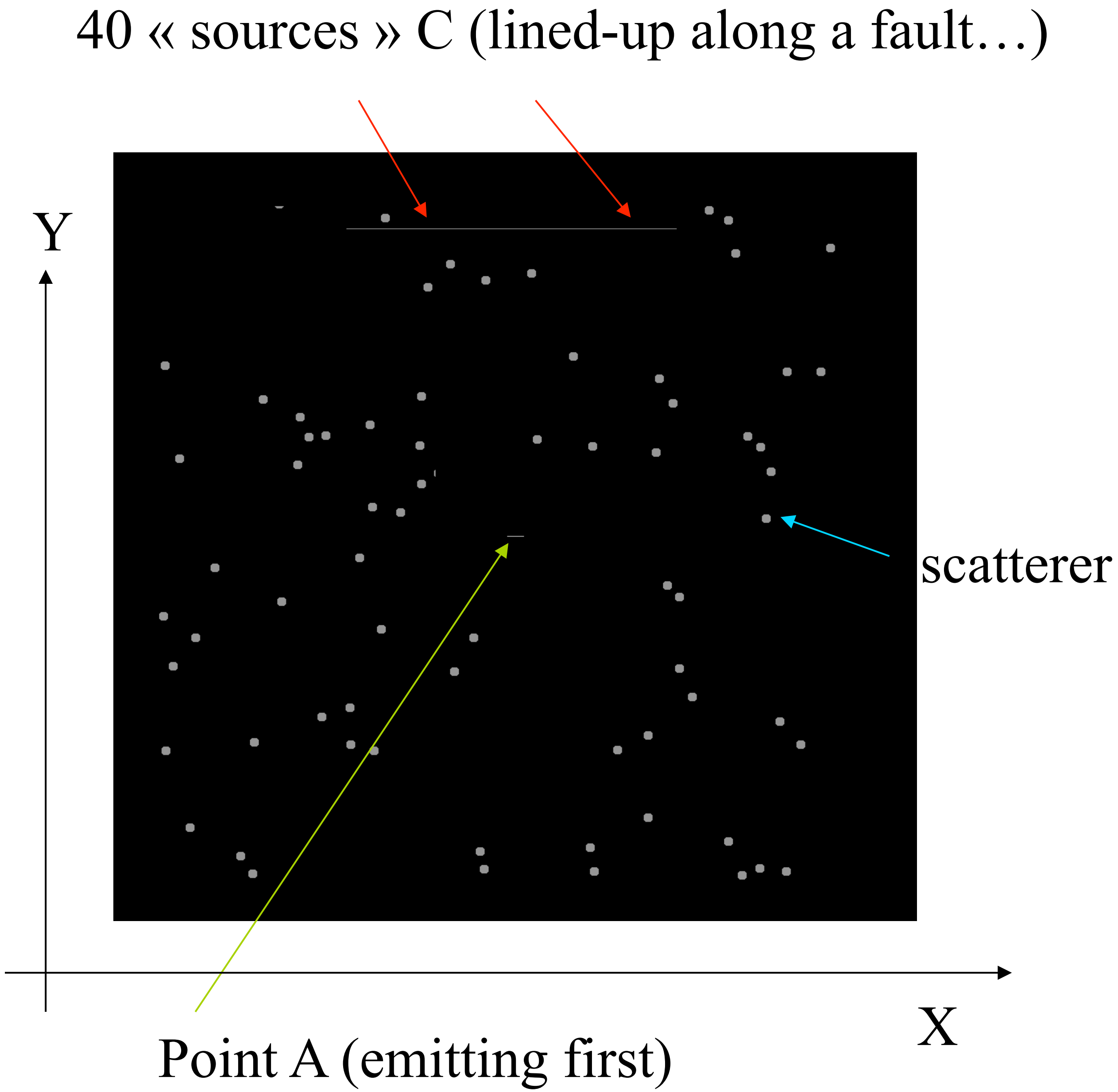


Nearly perfect refocalisation

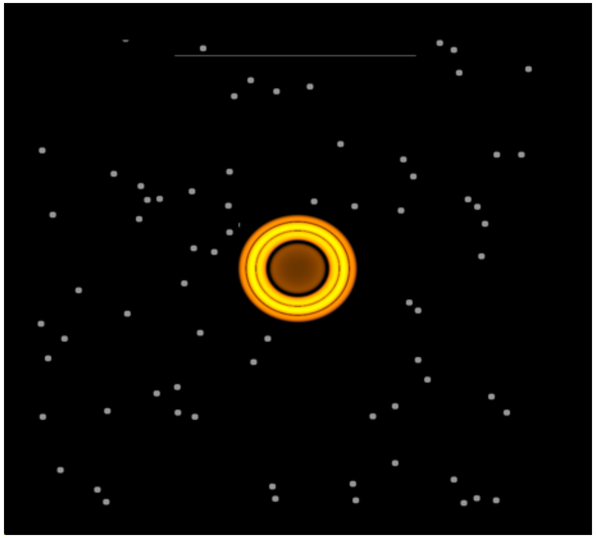
Re-emission from A :
G (t)



A more realistic configuration of sources

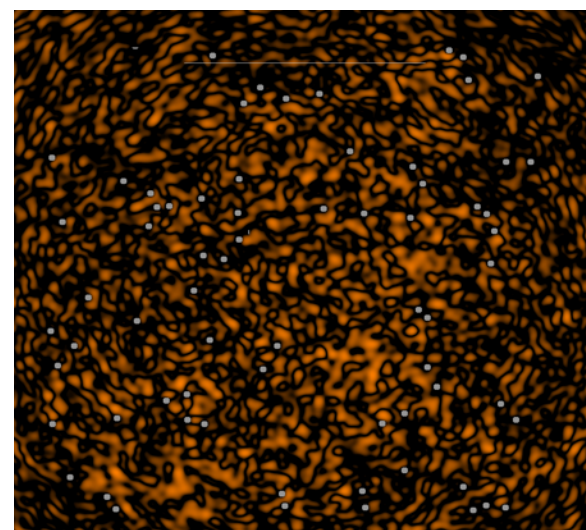


Time reversal experiment

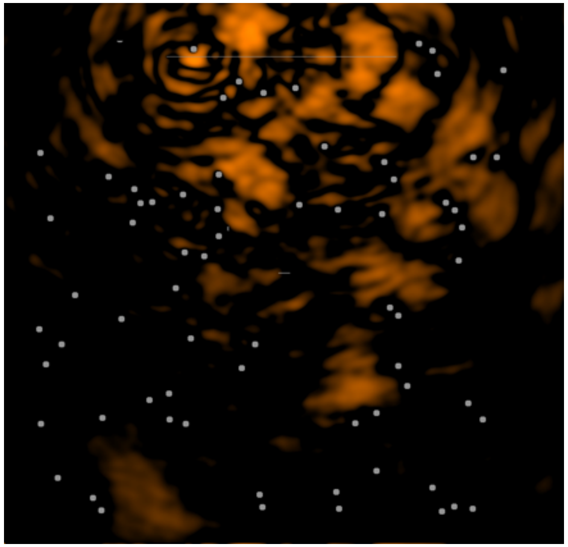


A send a pulse

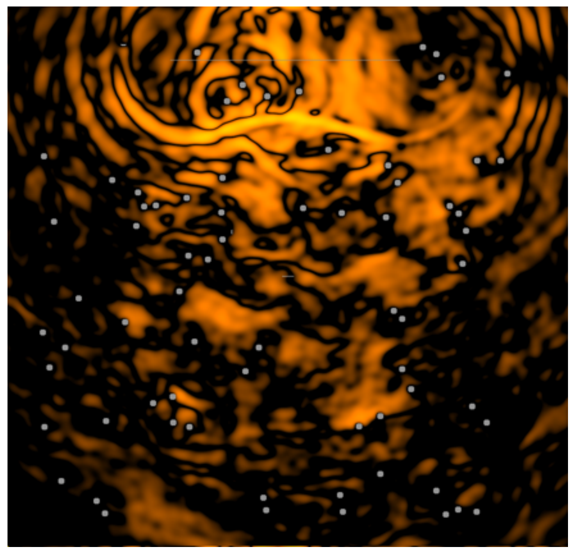
Scattering effects



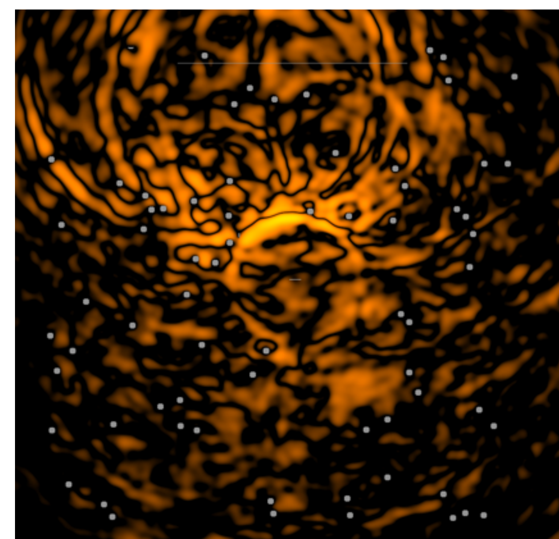
Diffuse field is
also recorded



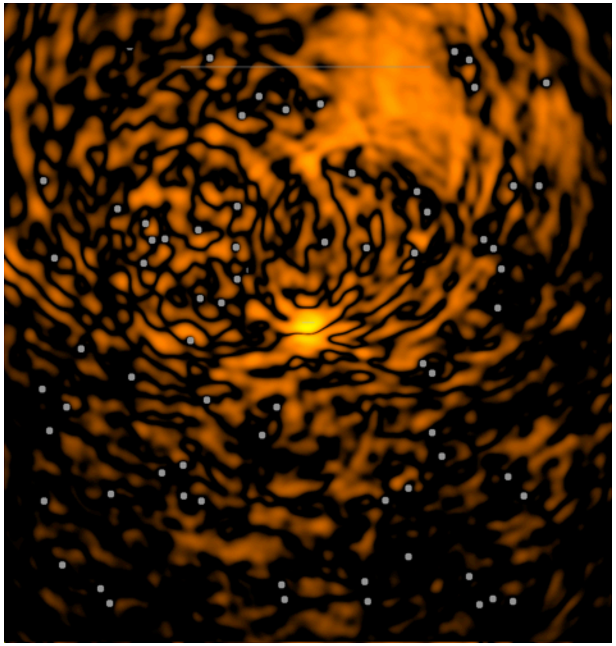
Re-emission



Converging field : G(-t)

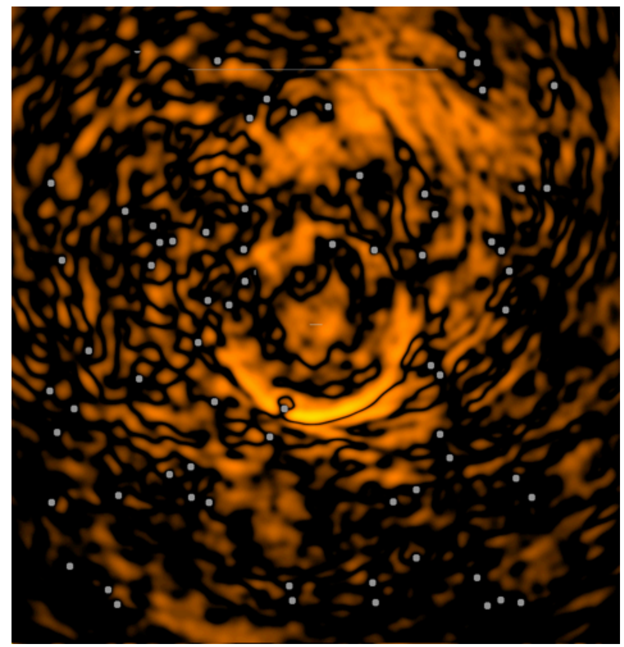


time



Partial focalisation

Diverging field : $h_{AB}(t)$



The symmetry of the Green function is lost!