Fluid Dynamics Research

PAPER

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To cite this article: Rodion Stepanov and Franck Plunian 2018 *Fluid Dyn. Res.* **50** 051409

IOP Publishing | The Japan Society of Fluid Mechanics Fluid *Dynamics* Research

Fluid Dyn. Res. **50** (2018) 051409 (9pp) https://doi.org/10.1088/1873-7005/aab440

Kinematic dynamo in a tetrahedron of Fourier modes

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Received 1 December 2017, revised 15 February 2018
Accepted for publication 6 March 2018
Published 9 August 2018

CrossMark

Communicated by Morten Brons

Abstract

Considering low-order systems of kinematic dynamo, we look for the lowest possible system order which can lead to a dynamo instability. For that we decompose, in Fourier space, both velocity and magnetic fields into helical modes. Starting with a single triad, which is the lowest possible order system, we show that a dynamo instability cannot occur, unless both velocity and magnetic fields are decimated. Decimation means that only one helical mode per wave number is kept, which is unlikely in a physically realizable situation. The next possible system order is the one composed of a set of four triads forming a tetrahedron. In that case we show that a dynamo instability is possible, without needing to decimate either the velocity field or the magnetic field. Finally we find that dynamo action is not possible if the kinetic helicity is zero at each wave number.

Keywords: kinematic dynamo, helical mode decomposition, low dimensional model, antidynamo theorem

(Some figures may appear in colour only in the online journal)

1. Introduction

Dynamo action is the physical phenomenon corresponding to a magnetic instability produced by the motion of an electrically conducting fluid. The time evolution of the magnetic field **b** obeys to the induction equation

$$(\partial_t - \eta\nabla^2)\mathbf{b} = \nabla \times (\mathbf{u} \times \mathbf{b}), \tag{1}$$

where $\mathbf{u}$ is the velocity field and η the magnetic diffusivity. We speak of a kinematic dynamo when the dynamic feedback of the magnetic field onto the flow is ignored. In that case the velocity field is considered as prescribed.

Similarly to what has been done for convection instability, it is tempting to derive low-order systems composed of only a few velocity and magnetic modes (Verma *et al* 2008, Kumar and Wahi 2017). Of course such models cannot account for true dynamos as by construction they are composed of only a limited number of magnetic scales. As a result magnetic diffusion, which acts against the dynamo instability and is increasing as the square of the wave number, is most probably underestimated. However such models present dynamic behaviors comparable to natural dynamos while authorizing a detailed description of the dynamo mechanisms in action.

Following the spirit of low-order systems, the present study aims at finding a kinematic dynamo model of the lowest order. In order to carry the analytical analysis as far as we can, we first project all quantities and induction equation on a Fourier basis. Working within triads guarantees the conservation of all ideally conserved quantities (e.g. kinetic energy and helicity in three dimensional hydrodynamics). Then we decompose all Fourier modes on a basis of helical modes, leading to a tractable system of equations for the diffusionless induction equation. We consider two cases of increasing complexity, first within a single triad (section 3) and second within four triads forming a tetrahedron (section 4). As we look for a magnetic instability, a negative or zero growth rate in absence of magnetic diffusivity ($\eta = 0$) is sufficient to conclude for the impossibility of dynamo action. On the other hand, a positive growth rate with $\eta = 0$ does not necessarily mean dynamo action, then needing a numerical confirmation (section 4).

2. Helical decomposition in Fourier space

2.1. Fourier space

In real space kinetic energy and helicity are defined as follows

$$E(\mathbf{x}) = \frac{1}{2}\mathbf{u}^2, \quad H(\mathbf{x}) = \frac{1}{2}\mathbf{u} \cdot \nabla \times \mathbf{u}. \tag{2}$$

We also introduce a potential kinetic helicity defined as

$$F(\mathbf{x}) = \frac{1}{2}\mathbf{w} \cdot \nabla \times \mathbf{w}, \tag{3}$$

where $\mathbf{w}$ is a velocity potential vector such that $\mathbf{u} = \nabla \times \mathbf{w}$.

Assuming triply periodic boundary conditions in a cube, both velocity and magnetic fields can be expanded into discrete Fourier series

$$\mathbf{u}(\mathbf{x}) = \sum_{\mathbf{k}} \mathbf{u}(\mathbf{k})\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}, \qquad \mathbf{b}(\mathbf{x}) = \sum_{\mathbf{k}} \mathbf{b}(\mathbf{k})\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}, \tag{4}$$

where $\mathbf{u}(\mathbf{k})$ and $\mathbf{b}(\mathbf{k})$ are complex Fourier coefficients. Then the induction equation (1) takes the form

$$(d_t + \eta k^2)\mathbf{b}(\mathbf{k}) = \sum_{\mathbf{k}+\mathbf{p}+\mathbf{q}=0} \mathrm{i}\,\mathbf{k} \times (\mathbf{u}^*(\mathbf{p}) \times \mathbf{b}^*(\mathbf{q})), \tag{5}$$

where the sum is a double sum on all $\mathbf{p}$ and $\mathbf{q}$ defining the triads $(\mathbf{k}, \mathbf{p}, \mathbf{q})$ such that $\mathbf{k} + \mathbf{p} + \mathbf{q} = 0$.

The kinetic energy and kinetic helicity take the form

$$E(\mathbf{k}) = \frac{1}{2}\mathbf{u}(\mathbf{k})\mathbf{u}^*(\mathbf{k}), \quad H(\mathbf{k}) = \frac{1}{2}(\mathrm{i}\mathbf{k},\, \mathbf{u}(\mathbf{k}),\, \mathbf{u}^*(\mathbf{k})). \tag{6}$$

The potential kinetic helicity satisfies

$$F(\mathbf{k}) = k^{-2}H(\mathbf{k}). \tag{7}$$

The advantage of working within the framework of triads is that all quantities which must be conserved in absence of viscosity and magnetic diffusivity are automatically conserved within each single triad. At the onset of dynamo action the kinetic energy and helicity are conserved. After the onset the conserved quantities are the total energy (magnetic plus kinetic), the magnetic helicity and the cross helicity (Plunian *et al* 2013).

2.2. Helical decomposition

Following the approach introduced by Craya (1958), Herring (1974), Cambon and Jacquin (1989), Waleffe (1992) we consider a basis of polarized helical waves $\mathbf{h}^{\pm}$ defined as the eigenvectors of the curl operator,

$$\mathrm{i}\mathbf{k} \times \mathbf{h}^{\pm}(\mathbf{k}) = \pm k\mathbf{h}^{\pm}(\mathbf{k}). \tag{8}$$

The Fourier modes of velocity and magnetic fields are then expanded on that helical basis

$$\mathbf{u}(\mathbf{k}) = u^{+}(\mathbf{k})\mathbf{h}^{+}(\mathbf{k}) + u^{-}(\mathbf{k})\mathbf{h}^{-}(\mathbf{k}), \tag{9}$$

$$\mathbf{b}(\mathbf{k}) = b^{+}(\mathbf{k})\mathbf{h}^{+}(\mathbf{k}) + b^{-}(\mathbf{k})\mathbf{h}^{-}(\mathbf{k}). \tag{10}$$

We note that the helical vectors $\mathbf{h}^{\pm}(\mathbf{k})$ are defined up to an arbitrary rotation of axis $\mathbf{k}$. Taking

$$\mathbf{h}^{\pm}(\mathbf{k}) = \mathbf{u}_2(\mathbf{k}) \pm \mathrm{i}\mathbf{u}_1(\mathbf{k}) \tag{11}$$

with $\mathbf{u}_1(\mathbf{k}) = (\mathbf{z}_\mathbf{k} \times \mathbf{k})/|(\mathbf{z}_\mathbf{k} \times \mathbf{k})|$ and $\mathbf{u}_2(\mathbf{k}) = \mathbf{u}_1(\mathbf{k}) \times \mathbf{k}/k$, where $\mathbf{z}_\mathbf{k}$ is an arbitrary vector that, in general, may depend on $\mathbf{k}$, though not proportional to $\mathbf{k}$, the kinetic energy, kinetic helicity and potential kinetic helicity at wave number $\mathbf{k}$ take the form

$$E(\mathbf{k}) = \frac{1}{2}(|u^{+}(\mathbf{k})|^2 + |u^{-}(\mathbf{k})|^2), \tag{12}$$

$$H(\mathbf{k}) = \frac{k}{2}(|u^{+}(\mathbf{k})|^2 - |u^{-}(\mathbf{k})|^2), \tag{13}$$

$$F(\mathbf{k}) = \frac{1}{2k}(|u^{+}(\mathbf{k})|^2 - |u^{-}(\mathbf{k})|^2). \tag{14}$$

Such a helical decomposition has recently been brought up to date. It is indeed an interesting tool to discriminate between the two types of helicity, negative or positive and study their interactions in hydrodynamics (Lessinnes *et al* 2011, Biferale *et al* 2012, Kessar *et al* 2015, Stepanov *et al* 2015, Alexakis 2016) and magnetohydrodynamics (Lessinnes *et al* 2009, Linkmann *et al* 2016).

Replacing the expressions (9), (10) of $\mathbf{u}$ and $\mathbf{b}$ in equation (5), and projecting onto the helical basis $\mathbf{h}^{s_k}(\mathbf{k})$ (with $s_k = \pm 1$) leads to the following form of the induction equation (Lessinnes *et al* 2009)

$$(d_t + \eta k^2)b^{s_k} = -s_k\, k \sum_{\Delta_{\mathbf{kpq}}} \sum_{s_p, s_q = \pm 1} g_{s_k, s_p, s_q}[u^{s_p}b^{s_q} - b^{s_p}u^{s_q}]^*, \tag{15}$$

where the first sum is made over all possible triads $\Delta_{\mathbf{kpq}}$, meaning over all possible triads $(\mathbf{k},\, \mathbf{p},\, \mathbf{q})$ satisfying $\mathbf{k} + \mathbf{p} + \mathbf{q} = 0$. In particular the sum is not repeated when exchanging $\mathbf{p}$

and $\mathbf{q}$, contrary to (5). From now and in the rest of the paper u^{s_k}, u^{s_p}, u^{s_q}, b^{s_k}, b^{s_p}, b^{s_q} stand for $u^{s_k}(\mathbf{k})$, $u^{s_p}(\mathbf{p})$, $u^{s_q}(\mathbf{q})$, $b^{s_k}(\mathbf{k})$, $b^{s_p}(\mathbf{p})$, $b^{s_q}(\mathbf{q})$, and g_{s_k,s_p,s_q} denotes a function of $\mathbf{k}$, $\mathbf{p}$, $\mathbf{q}$, s_k,s_p,s_q which is defined as

$$g_{s_k,s_p,s_q} = -\frac{(\mathbf{h}^{s_k}(\mathbf{k}),\, \mathbf{h}^{s_p}(\mathbf{p}),\, \mathbf{h}^{s_q}(\mathbf{q}))^*}{\mathbf{h}^{s_k}(\mathbf{k})^* \cdot \mathbf{h}^{s_k}(\mathbf{k})}. \tag{16}$$

Considering a single triadic interaction, it is not necessary to introduce an arbitrary unit vector $\mathbf{z_k}$ to define the unit vectors $\mathbf{u}_1$ and $\mathbf{u}_2$. Indeed, there is a natural direction which is represented by the unit vector perpendicular to the plane of the triad:

$$\boldsymbol{\lambda} = (\mathbf{k} \times \mathbf{p})/|\mathbf{k} \times \mathbf{p}| = (\mathbf{p} \times \mathbf{q})/|\mathbf{p} \times \mathbf{q}| = (\mathbf{q} \times \mathbf{k})/|\mathbf{q} \times \mathbf{k}|. \tag{17}$$

A second unit vector $\boldsymbol{\mu}_\mathbf{k} = \mathbf{k} \times \boldsymbol{\lambda}/k$ can be introduced and the helical vectors are then defined as

$$\mathbf{h}^{s_k}(\mathbf{k}) = \mathrm{e}^{\mathrm{i} s_k \varphi_\mathbf{k}}\, (\boldsymbol{\lambda} + \mathrm{i}\, s_k\, \boldsymbol{\mu}_\mathbf{k}). \tag{18}$$

The angle $\varphi_\mathbf{k}$ defines the rotation around $\mathbf{k}$ needed to transform the basis $(\boldsymbol{\mu}_\mathbf{k}, \boldsymbol{\lambda})$ into the basis $(\mathbf{u}_1(\mathbf{k}), \mathbf{u}_2(\mathbf{k}))$. Since the basis $(\boldsymbol{\mu}_\mathbf{k}, \boldsymbol{\lambda})$ depends on the triad, the angle $\varphi_\mathbf{k}$ is also a function of $(\mathbf{k}, \mathbf{p}, \mathbf{q})$. The coupling constant for this triad then simply reduces to

$$g_{s_k,s_p,s_q} = -\frac{1}{2}\mathrm{e}^{-\mathrm{i}(s_k\varphi_k + s_p\varphi_p + s_q\varphi_q)}\, s_k\, s_p\, s_q\, (s_k \sin\alpha_k + s_p \sin\alpha_p + s_q \sin\alpha_q) \tag{19}$$

with

$$\sin\alpha_k = \frac{Q_{kpq}}{2\,p\,q},\ \sin\alpha_p = \frac{Q_{kpq}}{2\,k\,q},\ \sin\alpha_q = \frac{Q_{kpq}}{2\,k\,p}, \tag{20}$$

and $Q_{kpq} = \sqrt{2\,k^2\,p^2 + 2\,q^2\,p^2 + 2\,q^2\,k^2 - k^4 - q^4 - p^4}$ which, after Heron's formula, is four times the surface of the triad $\Delta_\mathbf{kpq}$.

3. Solution in one triad

After (15) the diffusionless ($\eta = 0$) induction equation within a single triad $\Delta_\mathbf{kpq}$ is given by

$$\begin{aligned}
d_t b_k^+ &= -k[g_{+,+,+}(u_p^+ b_q^+ - b_p^+ u_q^+)^* + g_{+,-,-}(u_p^- b_q^- - b_p^- u_q^-)^*\\
&\quad + g_{+,+,-}(u_p^+ b_q^- - b_p^+ u_q^-)^* + g_{+,-,+}(u_p^- b_q^+ - b_p^- u_q^+)^*],\\
d_t b_p^+ &= -p[g_{+,+,+}(u_q^+ b_k^+ - b_q^+ u_k^+)^* + g_{-,+,-}(u_q^- b_k^- - b_q^- u_k^-)^*\\
&\quad + g_{-,+,+}(u_q^+ b_k^- - b_q^+ u_k^-)^* + g_{+,+,-}(u_q^- b_k^+ - b_q^- u_k^+)^*],\\
d_t b_q^+ &= -q[g_{+,+,+}(u_k^+ b_p^+ - b_k^+ u_p^+)^* + g_{-,-,+}(u_k^- b_p^- - b_k^- u_p^-)^*\\
&\quad + g_{+,-,+}(u_k^+ b_p^- - b_k^+ u_p^-)^* + g_{-,+,+}(u_k^- b_p^+ - b_k^- u_p^+)^*],\\
d_t b_k^- &= +k[g_{-,+,+}(u_p^+ b_q^+ - b_p^+ u_q^+)^* + g_{-,-,-}(u_p^- b_q^- - b_p^- u_q^-)^*\\
&\quad + g_{-,+,-}(u_p^+ b_q^- - b_p^+ u_q^-)^* + g_{-,-,+}(u_p^- b_q^+ - b_p^- u_q^+)^*],\\
d_t b_p^- &= +p[g_{+,-,+}(u_q^+ b_k^+ - b_q^+ u_k^+)^* + g_{-,-,-}(u_q^- b_k^- - b_q^- u_k^-)^*\\
&\quad + g_{-,-,+}(u_q^+ b_k^- - b_q^+ u_k^-)^* + g_{+,-,-}(u_q^- b_k^+ - b_q^- u_k^+)^*],\\
d_t b_q^- &= +q[g_{+,+,-}(u_k^+ b_p^+ - b_k^+ u_p^+)^* + g_{-,-,-}(u_k^- b_p^- - b_k^- u_p^-)^*\\
&\quad + g_{+,-,-}(u_k^+ b_p^- - b_k^+ u_p^-)^* + g_{-,+,-}(u_k^- b_p^+ - b_k^- u_p^+)^*].
\end{aligned} \tag{21}$$

The system (21) can be written in the form $d_t X = AX^*$, where A is a 6-by-6 matrix and $X = (b_k^+, b_p^+, b_q^+, b_k^-, b_p^-, b_q^-)$. The second time derivative of X obeys to $d_t^2 X = AA^* X$. Then calculating the set of complex eigenvalues $a_{i=1,6}$, of AA^*, the growth rate of X will correspond to $\Re\{a^{1/2}\} = \max_{i=1,6}(\Re\{a_i^{1/2}\})$. If a is real and negative then X will be oscillatory or stationary, without growth ($\Re\{a^{1/2}\} = 0$), ruling out any hope of dynamo action for non zero magnetic diffusivity. In all other cases dynamo action may be possible depending on the level of magnetic diffusivity.

3.1. One full triad

Choosing $\mathbf{z_k} = \mathbf{z_p} = \mathbf{z_q} = \boldsymbol{\lambda}$, we find that the non trivial eigenvalues of (21) correspond to the square roots of

$$a = -\frac{Q_{kpq}^2}{4}\left(\left|\frac{u_k^+ - u_k^-}{k}\right|^2 + \left|\frac{u_p^+ - u_p^-}{p}\right|^2 + \left|\frac{u_q^+ - u_q^-}{q}\right|^2\right). \tag{22}$$

As $\Re\{a^{1/2}\} = 0$, this shows that one triad alone cannot lead to dynamo action for non zero magnetic diffusivity.

The same conclusion can also been found directly from (1) choosing a system of Cartesian coordinates $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ with $\mathbf{z} = \boldsymbol{\lambda}$. In that case both velocity and magnetic fields depend on $\mathbf{x}$ and $\mathbf{y}$ only (and not on $\mathbf{z}$). Then the $\mathbf{x}$ and $\mathbf{y}$ components of the magnetic field decouple from the $\mathbf{z}$-component of the flow (and magnetic field), having no other option than decaying, as already shown by Zel'dovich and Ruzmaikin (1980) for a two-dimensional motion. Therefore to obtain dynamo action it is necessary to involve an additional magnetic field mode which does not belong to the plane $\Delta_{\mathbf{kpq}}$ as will be done in section 4.

3.2. One helical triad

It is tempting to consider the case of a single helical triad corresponding to only one single helical mode per wave number, setting $b^{-s_k} = b^{-s_p} = b^{-s_q} = u^{-s_k} = u^{-s_p} = u^{-s_q} = 0$. Following (Biferale *et al* 2012) we speak of a decimated system of equations. We emphasize that the ideally conserved quantities mentioned in section 2.1 are also conserved within each single helical triad.

The system (21) reduces to three equations and three unknowns

$$\begin{aligned} d_t b^{s_k} &= -s_k k g_{s_k,s_p,s_q}(u^{s_p} b^{s_q} - b^{s_p} u^{s_q})^*, \\ d_t b^{s_p} &= -s_p p g_{s_k,s_p,s_q}(u^{s_q} b^{s_k} - b^{s_q} u^{s_k})^*, \\ d_t b^{s_q} &= -s_q q g_{s_k,s_p,s_q}(u^{s_k} b^{s_p} - b^{s_k} u^{s_p})^*, \end{aligned} \tag{23}$$

with non trivial eigenvalues corresponding to the square roots of

$$a = -2 s_k s_p s_q kpq |g_{s_k,s_p,s_q}|^2 F(\Delta_{\mathbf{kpq}}), \tag{24}$$

where $F(\Delta_{\mathbf{kpq}})$ is the potential kinetic helicity for the whole triad $\Delta_{\mathbf{kpq}}$, given by

$$F(\Delta_{\mathbf{kpq}}) = F(\mathbf{k}) + F(\mathbf{p}) + F(\mathbf{q}) \tag{25}$$

$$= s_k \frac{|u^{s_k}|^2}{2k} + s_p \frac{|u^{s_p}|^2}{2p} + s_q \frac{|u^{s_q}|^2}{2q}. \tag{26}$$

We find that if $s_k = s_p = s_q$ then a is negative suggesting that, within a triad of only one type of helical modes, dynamo action cannot occur. On the other hand taking $u^{s_k} = 0$ and $s_k = -s_p = -s_q$ leads to positive a. Then there is always a magnetic field scale k with helicity opposite to u such that dynamo action occurs. The importance of such potential helicity has already been put forward in the context of large scale dynamos (Rädler and Brandenburg 2008) with indeed opposite helical modes between the large scale magnetic field and the flow.

We note that the antidynamo theorem that was invoked in section 3.1 in the case of a full triad does not apply to the case of an helical triad. This suggests that arbitrary decimation may change the physics of the dynamo instability. Therefore such decimation should be used with caution, unless it naturally arise from an external input, like for example an imposed external magnetic field.

4. Solution in one tetrahedron

In addition to wave vectors $\mathbf{k}$, $\mathbf{p}$, $\mathbf{q}$ we now introduce an additional wave vector $\mathbf{q}'$ which is not coplanar with $\Delta_{\mathbf{kpq}}$. The resulting tetrahedron is then formed of four triads ($\mathbf{k}$, $\mathbf{p}$, $\mathbf{q}$), ($\mathbf{k}$, $\mathbf{p}'$, $\mathbf{q}'$), ($\mathbf{p}$, $\mathbf{q}'$, $\mathbf{k}'$) and ($\mathbf{q}$, $\mathbf{k}'$, $\mathbf{p}'$) as shown in figure 1.

In order to keep the problem analytically tractable we assume that the flow is given by the two vectors $\mathbf{u_k}$ and $\mathbf{u_p}$ only, setting $u_q^{\pm} = u_{k'}^{\pm} = u_{p'}^{\pm} = u_{q'}^{\pm} = 0$. The resulting diffusionless problem corresponds to a 12-by-12 system with non trivial eigenvalues corresponding to the square roots of

$$a = -\frac{Q_{kp'q'}^2}{4}\left|\frac{u_k^+ - u_k^-}{k}\right|^2 - \frac{Q_{pq'k'}^2}{4}\left|\frac{u_p^+ - u_p^-}{p}\right|^2 \pm Q_{kp'q'}Q_{pq'k'}\frac{|\sin\psi_{q'}|}{q'}\sqrt{F(\mathbf{k})F(\mathbf{p})}, \tag{27}$$

where $\psi_{q'}$ is the angle between the planes defined by the triads ($\mathbf{k}$, $\mathbf{p}'$, $\mathbf{q}'$) and ($\mathbf{p}$, $\mathbf{q}'$, $\mathbf{k}'$). If the two triads are coplanar, corresponding to $\sin\psi_{q'} = 0$, then a is always negative, ruling out any possibility for dynamo action. We immediately see from (27) that it is the product $F(\mathbf{k})F(\mathbf{p})$ which is important for the dynamo action. If $F(\mathbf{k})F(\mathbf{p}) = 0$ then only decaying solutions are possible. Dynamo action is possible only if $F(\mathbf{k})F(\mathbf{p}) \neq 0$ and for q' sufficiently small in comparison with k and p.

We note that, as helicity $H(\mathbf{k})$ differs from potential helicity $F(\mathbf{k})$ by a factor k^2, having $F(\mathbf{k})F(\mathbf{p}) = 0$ is equivalent to have $H(\mathbf{k})H(\mathbf{p}) = 0$. In the next section 5 the discussion will be hold in terms of helicity, instead of potential helicity, without loss of generality.

To eventually conclude on the possibility of dynamo action in a tetrahedron, magnetic diffusion must be added to the problem, that can be handled only numerically. As an example we choose the tetrahedron formed by the three vectors $\mathbf{k} = (2, 0, 0)$, $\mathbf{p} = (0, 2, 0)$ and $\mathbf{q}' = (0, 0, 1)$. The velocity helical modes are chosen such that $|u^+(\mathbf{k})| = |u^+(\mathbf{p})| = 1$ and $|u^-(\mathbf{k})| = |u^-(\mathbf{p})| = 0$, which corresponds to a maximally helical flow. Replacing this values in (27) leads to $a = 2$ and therefore to a magnetic growth rate $\gamma_0 = \sqrt{2}$. The equation (15) is solved numerically for different values of η, with random complex phase for $u^+(\mathbf{k})$ and $u^+(\mathbf{p})$ for each run. The resulting magnetic growth rate γ is plotted versus η^{-1} (equivalent to the magnetic Reynolds number) in figure 2. In the diffusionless limit we find that $\lim_{\eta\to 0}\gamma = \gamma_0$. The onset of dynamo action at critical η^* corresponds to $\gamma(\eta^*) = 0$.

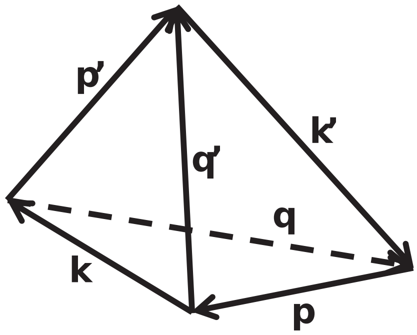

Figure 1. Tetrahedron configuration of four interacting triads $(\mathbf{k}, \mathbf{p}, \mathbf{q})$, $(\mathbf{k}, \mathbf{p}', \mathbf{q}')$, $(\mathbf{p}, \mathbf{q}', \mathbf{k}')$ and $(\mathbf{q}, \mathbf{k}', \mathbf{p}')$.

5. On the role of kinetic helicity in dynamo action

The role of kinetic helicity in dynamo action as been discussed for a long time starting with the mean-field approach, the expression of the so-called α-effect being directly proportional either to the kinetic helicity or to the potential kinetic helicity, depending on which limit is considered, either the high-conductivity or the low-conductivity limit (Krause and Rädler 1980). Since then, the possibility of dynamo action in non helical flows has been well documented, advocating either for time dependent flows (Gilbert *et al* 1988, Avalos-Zuniga *et al* 2009), potential helicity (Rädler and Brandenburg 2008) or memory effect (Rheinhardt *et al* 2014). Eventually direct numerical simulations of non helical turbulence lead to the generation of magnetic field by dynamo action (Kumar *et al* 2015, Kumar and Verma 2017) such that, today, kinetic helicity is known to be un-necessary for dynamo action.

However this is not what expression (27) tells us. Indeed, after (27) if at each wave number the kinetic helicity is zero $H(\mathbf{k}) = H(\mathbf{p}) = H(\mathbf{q}) = 0$, then dynamo action is found not to be possible. Of course if such a condition holds then it necessarily implies zero kinetic helicity for the whole triad $H(\Delta_{\mathbf{kpq}}) = 0$, as

$$H(\Delta_{\mathbf{kpq}}) = H(\mathbf{k}) + H(\mathbf{p}) + H(\mathbf{q}). \tag{28}$$

However the reciprocal is not true. This means that we can have zero kinetic helicity in each triad, leading to a flow with pointwise zero kinetic helicity, but non zero kinetic helicity at each wave number. This makes all the difference between our finding issued from (27) and what is usually understood as zero kinetic helicity.

To illustrate our point we consider two of the four flows introduced by Roberts (1972) because they present the two extreme possible cases with either maximal or zero helicity. The first flow $\mathbf{u}_1 = (\sin y, \sin x, \cos x - \cos y)$ is Beltrami ($\mathbf{u}_1 = \nabla \times \mathbf{u}_1$) and then maximally helical ($H_1(\mathbf{x}) = E_1(\mathbf{x})$) as in the numerical application of section 4. The second flow $\mathbf{u}_2 = (\sin y, \sin x, \cos x + \cos y)$ is pointwise zero helical ($H_2(\mathbf{x}) = 0$). Roberts (1972) showed that both flows were dynamo capable.

As these two flows depend only on two coordinates x and y we can apply our formula (27) in order to calculate a_1 and a_2. We take $\mathbf{k} = (1, 0, 0)$ and $\mathbf{p} = (0, 1, 0)$, leading to $\mathbf{u}_1(\mathbf{k}) = \left(0, -\frac{\mathrm{i}}{2}, \frac{1}{2}\right)$, $\mathbf{u}_1(\mathbf{p}) = \left(-\frac{\mathrm{i}}{2}, 0, -\frac{1}{2}\right)$ and $\mathbf{u}_2(\mathbf{k}) = \left(0, -\frac{\mathrm{i}}{2}, \frac{1}{2}\right)$, $\mathbf{u}_2(\mathbf{p}) = \left(-\frac{\mathrm{i}}{2}, 0, \frac{1}{2}\right)$. Choosing $\mathbf{h}^{s_k}(\mathbf{k}) = (0, -\mathrm{i}s_k, 1)$ and $\mathbf{h}^{s_p}(\mathbf{p}) = (\mathrm{i}s_p, 0, 1)$ we find that

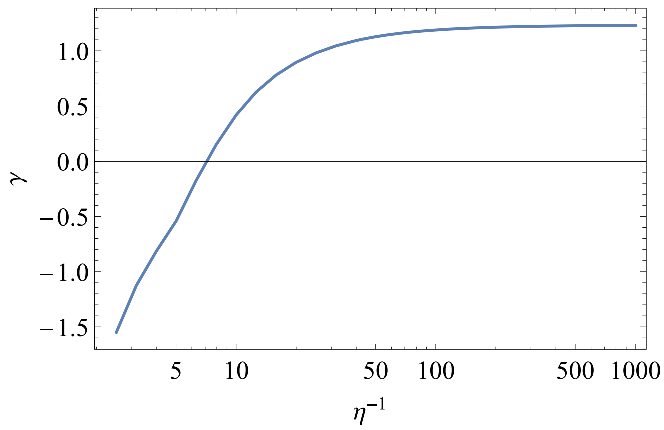

Figure 2. Plot of the magnetic field growth rate γ versus η^{-1}.

$$u_1^+(\mathbf{k}) = -u_1^+(\mathbf{p}) = \frac{1}{2}, \qquad u_1^-(\mathbf{k}) = u_1^-(\mathbf{p}) = 0, \tag{29}$$

$$u_2^+(\mathbf{k}) = u_2^-(\mathbf{p}) = \frac{1}{2}, \qquad u_2^-(\mathbf{k}) = u_2^+(\mathbf{p}) = 0. \tag{30}$$

Then for $\mathbf{q}' = (0, 0, q')$ we find

$$a_1 = \frac{1}{2}q'(-q' \pm 1), \quad a_2 = \frac{1}{2}q'(-q' \pm \mathrm{i}). \tag{31}$$

In the first case we can have $\Re\{a_1^{1/2}\} \geqslant 0$ provided $|q'| \leqslant 1$. In the second case whatever the value of q' among the possible square roots of a_2 there is always one such that $\Re(a_2^{1/2}) \geqslant 0$. Provided diffusion is neglected we find that both flows are dynamo capable. Now calculating the kinetic helicity for both flows we find that

$$H_1(\mathbf{k}) = H_1(\mathbf{p}) = \frac{1}{8}, \quad H_2(\mathbf{k}) = -H_2(\mathbf{p}) = \frac{1}{8} \tag{32}$$

leading to

$$H_1(\Delta_{\mathbf{kpq}}) = \frac{1}{4}, \quad H_2(\Delta_{\mathbf{kpq}}) = 0. \tag{33}$$

The flow $\mathbf{u}_2$ is a clear example of a dynamo capable flow with zero pointwise helicity. As stated above it is the fact that the helicity is non zero at each wave number which is in fact crucial for dynamo action.

6. Concluding remarks

In section 3.2 we found that dynamo action in a single triad was possible only if the magnetic and the velocity fields are decimated, keeping only one helical mode per wave vector. In another context such a wave number decimation has been suggested and experimented in a recent model of turbulence (Gürcan 2017). However we also found in section 3.1 that the dynamo action disappears as soon as both types of helical modes are included, stressing that decimation should be used with caution.

Considering four triads forming a tetrahedron we found in section 4 that dynamo action becomes possible, without any need for decimation. In addition, our findings suggest the following antidynamo theorem, that *a flow with zero kinetic helicity at each wave number cannot lead to dynamo action*. However our study being limited to only two flow wave numbers, additional investigation with more complex flows needs to be done to confirm this assertion.

Acknowledgments

This work was supported by joint project of the Russian Science Foundation (grand RSF-16-41-02012) and the Department of Science Technology of the Ministry of Science and Technology of the Republic of India (grant INT/RUS/RSF/3).

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