Imaging with intensity cross correlations and application to ghost imaging

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- General topic: correlation-based imaging with noise sources.
- What about intensity only measurements ?
- What about the role of scattering ?
- Particular application: Ghost imaging.

Scalar wave equation and Green's function

- In this talk, we consider the scalar wave model in $\mathbb{R}^3$:

$$\frac{1}{c^2(\vec{\boldsymbol{x}})}\frac{\partial^2 u}{\partial t^2} - \Delta_{\vec{\boldsymbol{x}}} u = n(t, \vec{\boldsymbol{x}})$$

$n(t, \vec{\boldsymbol{x}})$: source.
$c(\vec{\boldsymbol{x}})$: propagation speed (parameter of the medium), assumed to be constant outside a domain with compact support.

In the Fourier domain:

$$\hat{u}(\omega, \vec{\boldsymbol{x}}) = \int u(t, \vec{\boldsymbol{x}}) e^{i\omega t} dt$$

we have

$$\hat{u}(\omega, \vec{\boldsymbol{x}}) = \int \hat{G}(\omega, \vec{\boldsymbol{x}}, \vec{\boldsymbol{y}}) \hat{n}(\omega, \vec{\boldsymbol{y}}) d\vec{\boldsymbol{y}}$$

where the time-harmonic Green's function $\hat{G}(\omega, \vec{\boldsymbol{x}}, \vec{\boldsymbol{y}})$ is the solution of the Helmholtz equation

$$\Delta_{\vec{\boldsymbol{x}}} \hat{G} + \frac{\omega^2}{c^2(\vec{\boldsymbol{x}})} \hat{G} = -\delta(\vec{\boldsymbol{x}} - \vec{\boldsymbol{y}}),$$

with the Sommerfeld radiation condition ($c(\vec{\boldsymbol{x}}) = c_0$ at infinity):

$$\lim_{|\vec{\boldsymbol{x}}| \to \infty} |\vec{\boldsymbol{x}}| \Big(\frac{\vec{\boldsymbol{x}}}{|\vec{\boldsymbol{x}}|} \cdot \nabla_{\vec{\boldsymbol{x}}} - i \frac{\omega}{c_0} \Big) \hat{G}(\omega, \vec{\boldsymbol{x}}, \vec{\boldsymbol{y}}) = 0$$

Green's function estimation with ambient noise sources (1/3)

$$\frac{1}{c^2(\vec{\boldsymbol{x}})}\frac{\partial^2 u}{\partial t^2} - \Delta_{\vec{\boldsymbol{x}}} u = n(t, \vec{\boldsymbol{x}})$$

- Sources $n(t, \vec{\boldsymbol{x}})$: Gaussian random process, stationary in time, with mean zero and covariance

$$\big\langle n(t_1, \vec{\boldsymbol{y}}_1) n(t_2, \vec{\boldsymbol{y}}_2) \big\rangle = F(t_2 - t_1) K(\vec{\boldsymbol{y}}_1) \delta(\vec{\boldsymbol{y}}_1 - \vec{\boldsymbol{y}}_2)$$

$\langle \cdot \rangle$: statistical average.
The function $\hat{F}$ is the power spectral density of the sources.
The function K characterizes the spatial support of the sources.

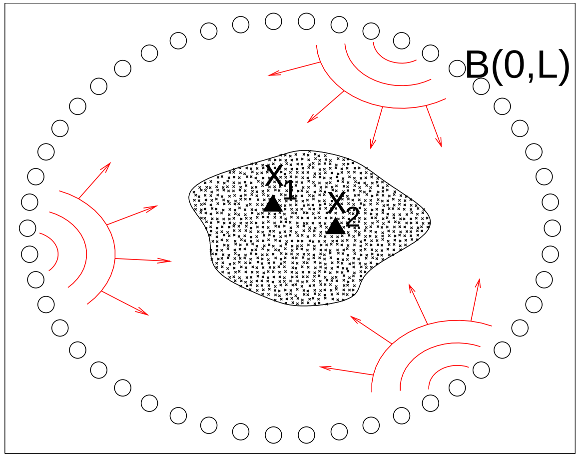
- The empirical cross correlation:

$$C_T(\tau, \vec{\boldsymbol{x}}_1, \vec{\boldsymbol{x}}_2) = \frac{1}{T}\int_0^T u(t, \vec{\boldsymbol{x}}_1) u(t+\tau, \vec{\boldsymbol{x}}_2) dt$$

converges in probability as $T \to \infty$ to the statistical cross correlation $C^{(1)}$ given by

$$\begin{aligned} C^{(1)}(\tau, \vec{\boldsymbol{x}}_1, \vec{\boldsymbol{x}}_2) &= \big\langle u(0, \vec{\boldsymbol{x}}_1) u(\tau, \vec{\boldsymbol{x}}_2) \big\rangle \\ &= \frac{1}{2\pi} \int d\vec{\boldsymbol{y}} \int d\omega \overline{\hat{G}}(\omega, \vec{\boldsymbol{x}}_1, \vec{\boldsymbol{y}}) \hat{G}(\omega, \vec{\boldsymbol{x}}_2, \vec{\boldsymbol{y}}) K(\vec{\boldsymbol{y}}) \hat{F}(\omega) e^{-i\omega\tau} \end{aligned}$$

Green's function estimation with ambient noise sources (2/3)



Cross correlation with noise sources distributed on a <u>closed surface</u> $\partial B(\mathbf{0}, L)$:

$$C^{(1)}(\tau, \vec{\boldsymbol{x}}_1, \vec{\boldsymbol{x}}_2) = \frac{1}{2\pi} \int d\omega \int_{\partial B(\mathbf{0},L)} d\sigma(\vec{\boldsymbol{y}}) \overline{\hat{G}}(\omega, \vec{\boldsymbol{x}}_1, \vec{\boldsymbol{y}}) \hat{G}(\omega, \vec{\boldsymbol{x}}_2, \vec{\boldsymbol{y}}) \hat{F}(\omega) e^{-i\omega\tau}$$

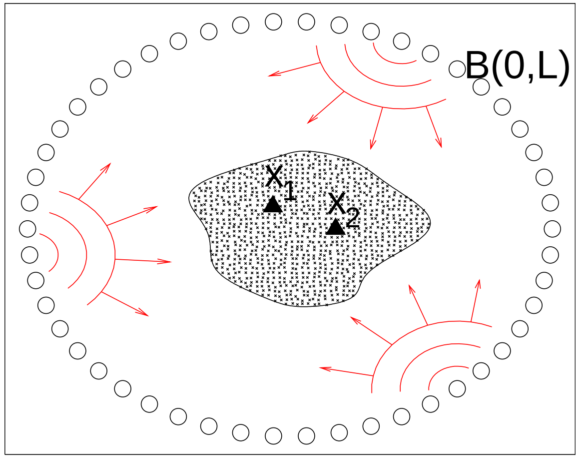
By Helmholtz-Kirchhoff identity,

$$\hat{G}(\omega, \vec{\boldsymbol{x}}_1, \vec{\boldsymbol{x}}_2) - \overline{\hat{G}}(\omega, \vec{\boldsymbol{x}}_1, \vec{\boldsymbol{x}}_2) = \frac{2i\omega}{c_0} \int_{\partial B(\mathbf{0},L)} d\sigma(\vec{\boldsymbol{y}}) \overline{\hat{G}}(\omega, \vec{\boldsymbol{x}}_1, \vec{\boldsymbol{y}}) \hat{G}(\omega, \vec{\boldsymbol{x}}_2, \vec{\boldsymbol{y}})$$

we have

$$C^{(1)}(\tau, \vec{\boldsymbol{x}}_1, \vec{\boldsymbol{x}}_2) = \frac{c_0}{4\pi} \int \frac{\hat{F}(\omega)}{\omega} \mathrm{Im}\big(\hat{G}(\omega, \vec{\boldsymbol{x}}_1, \vec{\boldsymbol{x}}_2)\big) e^{-i\omega\tau} d\omega$$

Green's function estimation with ambient noise sources (3/3)



$$\begin{aligned}\partial_\tau C^{(1)}(\tau,\vec{\boldsymbol{x}}_1,\vec{\boldsymbol{x}}_2) &= -\frac{ic_0}{4\pi}\int \hat{F}(\omega)\mathrm{Im}\big(\hat{G}(\omega,\vec{\boldsymbol{x}}_1,\vec{\boldsymbol{x}}_2)\big)e^{-i\omega\tau}d\omega \\ &= -\frac{c_0}{2}\Big(F *_\tau G(\tau,\vec{\boldsymbol{x}}_1,\vec{\boldsymbol{x}}_2) - F *_\tau G(-\tau,\vec{\boldsymbol{x}}_1,\vec{\boldsymbol{x}}_2)\Big)\end{aligned}$$

• The cross correlation of noise signals recorded by two passive sensors is related to the Green's function between the sensors.
↪ the passive sensors can be transformed into virtual sources.

• This result requires rather strong conditions on the noise sources (uniform distribution) [Schuster, CUP (2009), Wapenaar et al., Geophysics, 75 (2010)].
• Weaker results (for travel time estimation) can be obtained with weaker conditions [1].

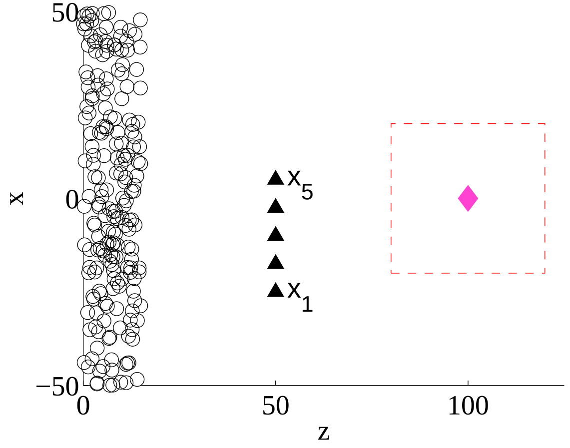
[1] J. Garnier and G. Papanicolaou, *SIAM J. Imaging Sciences* **2**, 396 (2009).

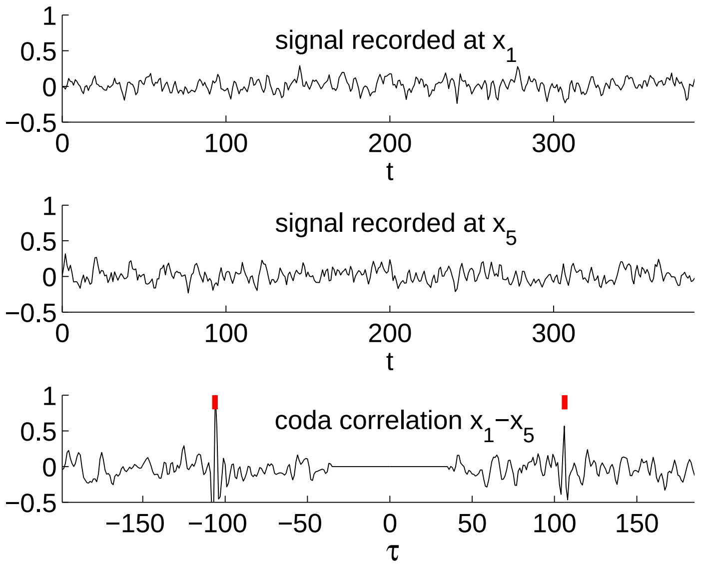
Reflector imaging with a passive receiver array

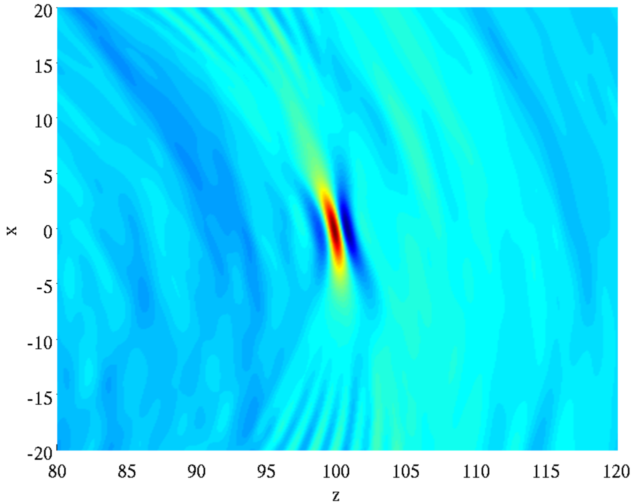
- Ambient noise sources (○) emit stationary random signals.
- The signals $(u(t,\vec{\boldsymbol{x}}_r))_{r=1,\dots,N_\mathrm{r}}$ are recorded by the receivers $(\vec{\boldsymbol{x}}_r)_{r=1,\dots,N_\mathrm{r}}$ (▲).
- The reflector (♦) is imaged by migration of the cross correlation matrix [1]:

$$\mathcal{I}(\vec{\boldsymbol{y}}^S) = \sum_{r,r'=1}^{N_\mathrm{r}} C_T\Big(\frac{|\vec{\boldsymbol{x}}_{r'}-\vec{\boldsymbol{y}}^S|}{c_0}+\frac{|\vec{\boldsymbol{x}}_{r}-\vec{\boldsymbol{y}}^S|}{c_0},\vec{\boldsymbol{x}}_r,\vec{\boldsymbol{x}}_{r'}\Big)$$

$$\text{with} \quad C_T(\tau,\vec{\boldsymbol{x}}_r,\vec{\boldsymbol{x}}_{r'}) = \frac{1}{T}\int_0^T u(t+\tau,\vec{\boldsymbol{x}}_{r'})u(t,\vec{\boldsymbol{x}}_r)dt$$







Good image provided the ambient noise illumination is long (in time) and diversified (in angle) [1].

[1] J. Garnier and G. Papanicolaou, *SIAM J. Imaging Sciences* **2**, 396 (2009).

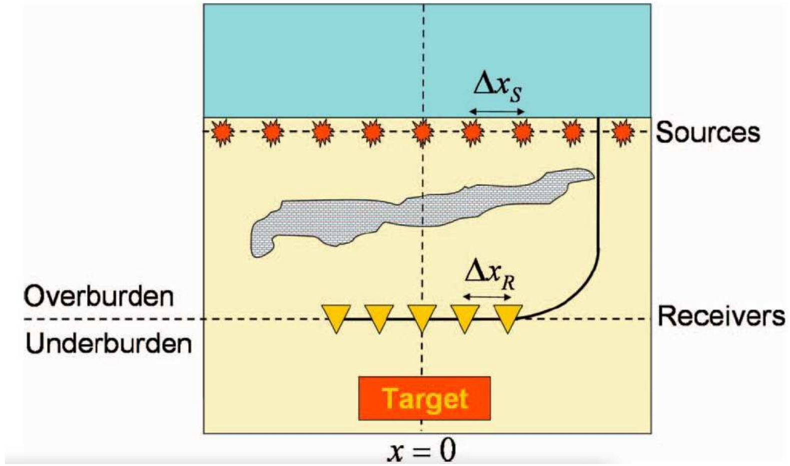
A successful application: seismic exploration below an overburden

Data: $\{u(t,\vec{\boldsymbol{x}}_r;\vec{\boldsymbol{x}}_s), t\in\mathbb{R}, r=1,\dots,N_{\rm r}, s=1,\dots,N_{\rm s}\}$

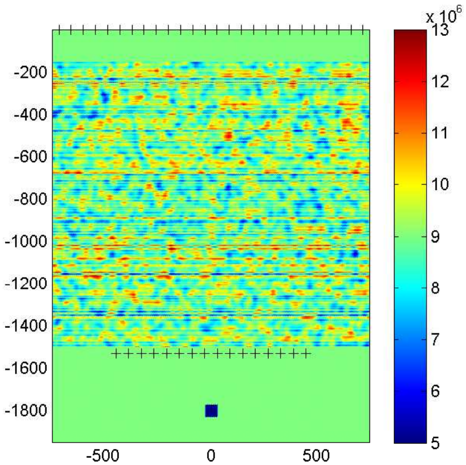
Correlations: $\{C(t,\vec{\boldsymbol{x}}_r,\vec{\boldsymbol{x}}_{r'}), t\in\mathbb{R}, r,r'=1,\dots,N_{\rm r}\}$

$C(t,\vec{\boldsymbol{x}}_r,\vec{\boldsymbol{x}}_{r'}) = \sum_{s=1}^{N_s}\int_{-\infty}^{\infty} u(t'+t,\vec{\boldsymbol{x}}_r;\vec{\boldsymbol{x}}_s)u(t',\vec{\boldsymbol{x}}_{r'};\vec{\boldsymbol{x}}_s)dt'$

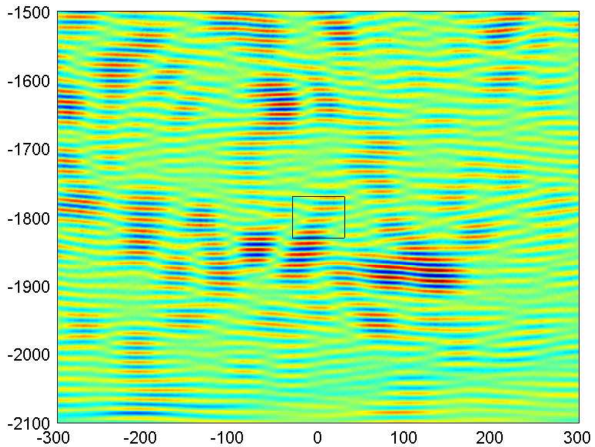
Imaging by migration of correlations [Bakulin and Calvert, Geophysics, 71 (2006)]



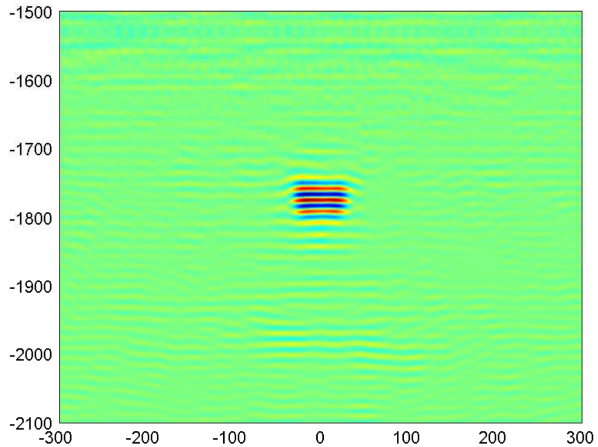
From Bakulin and Calvert



geometric set-up



data migration



correlation migration [1]

[1] J. Garnier et al., *Inverse Problems* **28**, 075002 (2012); *SIIMS* **7**, 1210 (2014); *SIIMS* **8**, 248 (2014).

What about intensity only measurements ?

• The analysis so far assumes that the recorded signals $(u(t))_{t\in\mathbb{R}}$ are time-resolved. This is OK in seismology and in acoustics (sampling rate > operating frequency).

• In optics, only intensities can be measured (time averages of the square of the wave field):

$$I(t) = \frac{1}{2T_e}\int_{-\infty}^{\infty} \Pi\Big(\frac{\tau}{T_e}\Big)u(t+\tau)^2 d\tau$$

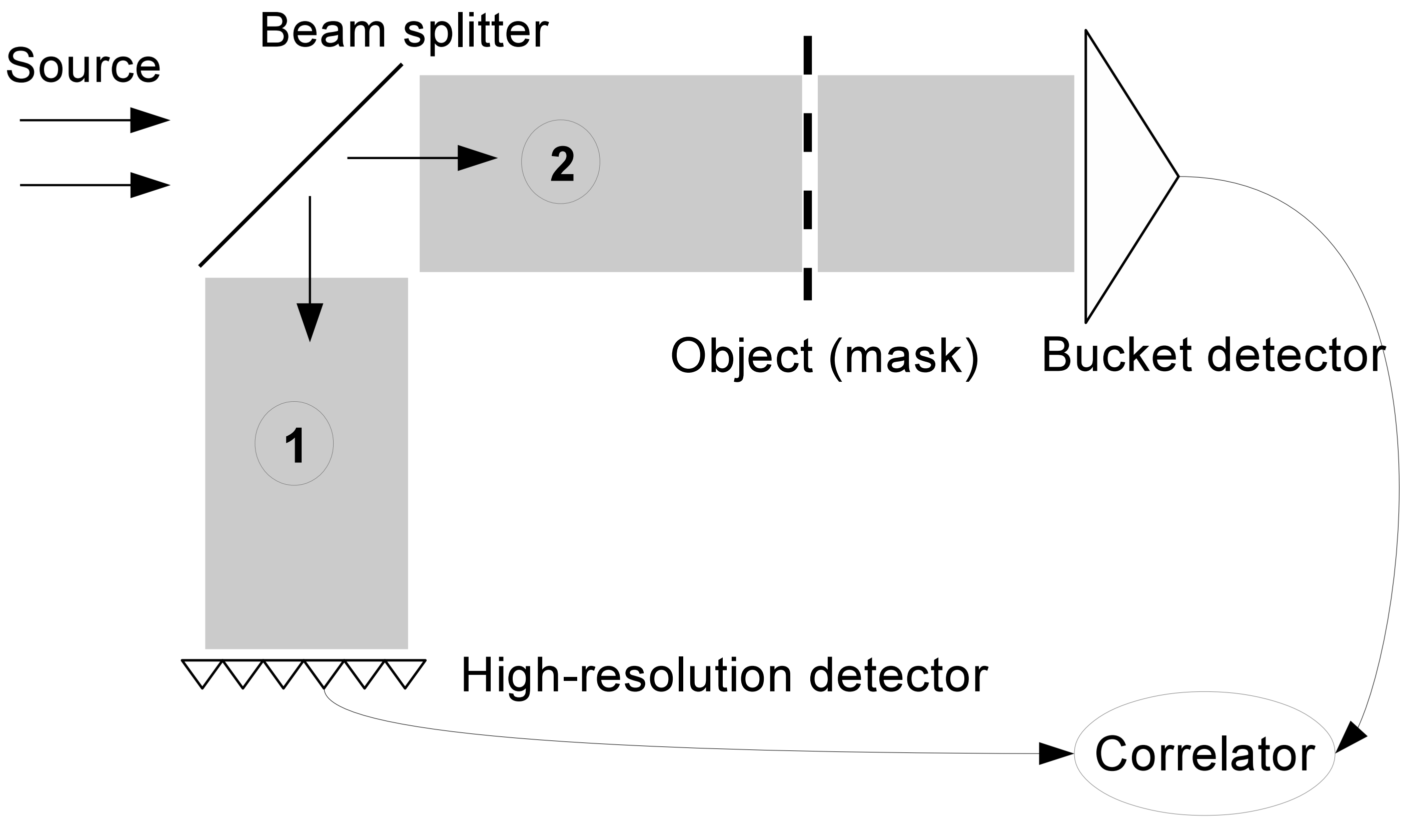
where T_e is the integration time of the sensor and Π is such that $\int \Pi(s)ds = 1$.
Assume that the wave field is:

$$u(t) = \exp\big(-i\omega_0 t\big)v(t) + c.c.,$$

where $c.c.$ stands for complex conjugate, ω_0 is the carrier frequency, and $v(t)$ is the complex-valued "slowly varying envelope", whose Fourier transform has a typical width B that is much smaller than ω_0.
If $\omega_0 T_e \gg 1 \gg BT_e$, then

$$I(t) \simeq |v(t)|^2.$$

Ghost imaging



- Noise source (laser light passed through a rotating glass diffuser).
- without object in path 1; a high-resolution detector measures the spatially-resolved intensity $I_1(t, \boldsymbol{x})$.
- with object (mask) in path 2; a single-pixel detector measures the spatially-integrated intensity $I_2(t)$.

Experimental result: the correlation of $I_1(\cdot, \boldsymbol{x})$ and $I_2(\cdot)$ is an image of the object [1,2].

[1] A. Valencia et al., *PRL* **94**, 063601 (2005); [2] J. H. Shapiro et al., *Quantum Inf. Process* **1** 949 (2012).

Ghost imaging

- Wave equation in paths 1 and 2:

$$\frac{1}{c_j^2(\vec{\boldsymbol{x}})}\frac{\partial^2 u_j}{\partial t^2} - \Delta_{\vec{\boldsymbol{x}}} u_j = e^{-i\omega_0 t} n(t,\boldsymbol{x})\delta(z) + c.c., \qquad \vec{\boldsymbol{x}} = (\boldsymbol{x}, z) \in \mathbb{R}^2 \times \mathbb{R}, \qquad j = 1,2$$

- Noise source:

$$\Big\langle n(t,\boldsymbol{x})\overline{n(t',\boldsymbol{x}')}\Big\rangle = F(t-t')\exp\Big(-\frac{|\boldsymbol{x}|^2}{r_0^2}\Big)\delta(\boldsymbol{x}-\boldsymbol{x}')$$

with the width of $\hat{F}(\omega)$ much smaller than ω_0.

- Wave fields:

$$u_j(t,\vec{\boldsymbol{x}}) = v_j(t,\vec{\boldsymbol{x}})e^{-i\omega_0 t} + c.c., \qquad j = 1,2$$

- Intensity measurements:

$$\begin{aligned} I_1(t,\boldsymbol{x}) &= |v_1(t,(\boldsymbol{x},L))|^2 \text{ in the plane of the high-resolution detector} \\ I_2(t) &= \int_{\mathbb{R}^2} |v_2(t,(\boldsymbol{x}',L+L_0))|^2 d\boldsymbol{x}' \text{ in the plane of the bucket detector} \end{aligned}$$

- Correlation:

$$C_T(\boldsymbol{x}) = \frac{1}{T}\int_0^T I_1(t,\boldsymbol{x})I_2(t)dt - \Big(\frac{1}{T}\int_0^T I_1(t,\boldsymbol{x})dt\Big)\Big(\frac{1}{T}\int_0^T I_2(t)dt\Big)$$

• Resolution analysis in homogeneous media:

$$\Big\langle n(t,\boldsymbol{x})\overline{n(t',\boldsymbol{x}')}\Big\rangle = F(t-t')\exp\Big(-\frac{|\boldsymbol{x}|^2}{r_0^2}\Big)\delta(\boldsymbol{x}-\boldsymbol{x}'),$$

Paraxial Green's function (valid when $\lambda_0 \ll r_0 \ll L$, with $\lambda_0 = 2\pi c_0/\omega_0$):

$$\hat{G}_0\big(\omega_0,(\boldsymbol{x},L),(\boldsymbol{y},0)\big) = \frac{1}{4\pi L}\exp\Big(i\frac{\omega_0}{c_0}L + i\frac{\omega_0}{c_0}\frac{|\boldsymbol{x}-\boldsymbol{y}|^2}{2L}\Big)$$

Model for the object: Mask $\mathcal{T}(\boldsymbol{x})$ in the plane $z=L$.

• Result:

$$C^{(1)}(\boldsymbol{x}) = \int_{\mathbb{R}^2} h(\boldsymbol{x}-\boldsymbol{r})|\mathcal{T}(\boldsymbol{r})|^2 d\boldsymbol{r}$$

with

$$h(\boldsymbol{x}) = \frac{r_0^4}{2^8\pi^2L^2}\exp\Big(-\frac{|\boldsymbol{x}|^2}{4\rho_{\rm gi0}^2}\Big), \qquad \rho_{\rm gi0}^2 = \frac{c_0^2L^2}{2\omega_0^2r_0^2}$$

Resolution: $\rho_{\rm gi0} \sim \lambda_0 L/r_0$ (Rayleigh resolution formula).

Sketch of ideal proof. Use the Gaussian summation rule (the fourth-order moments of Gaussian random fields can be expressed in terms of sums of products of second-order moments).

If $v(\boldsymbol{x})$ is a complex symmetric circular Gaussian random field, then

$$\mathrm{Cov}\big(|v(\boldsymbol{x})|^2,|v(\boldsymbol{x}')|^2\big) = \big|\mathrm{Cov}\big(v(\boldsymbol{x}),\overline{v(\boldsymbol{x}')}\big)\big|^2$$

- Extension for partially coherent source (Gauss-Schell model):

$$\left\langle n(t,\boldsymbol{x})\overline{n(t',\boldsymbol{x}')}\right\rangle = F(t-t')\exp\Big(-\frac{|\boldsymbol{x}+\boldsymbol{x}'|^2}{4r_0^2}-\frac{|\boldsymbol{x}-\boldsymbol{x}'|^2}{4\rho_0^2}\Big)$$

- Result:

$$C^{(1)}(\boldsymbol{x}) = \int_{\mathbb{R}^2} H(\boldsymbol{x},\boldsymbol{r})|\mathcal{T}(\boldsymbol{r})|^2 d\boldsymbol{r}$$

with

$$H(\boldsymbol{x},\boldsymbol{r}) = \frac{r_0^2\rho_0^2c_0^2}{64\omega_0^4\rho_{\rm gi1}^2R_{\rm gi1}^2}\exp\Big(-\frac{|\boldsymbol{x}-\boldsymbol{r}|^2}{4\rho_{\rm gi1}^2}-\frac{|\boldsymbol{x}+\boldsymbol{r}|^2}{4R_{\rm gi1}^2}\Big),$$

$$\rho_{\rm gi1}^2 = \rho_{\rm gi0}^2+\frac{\rho_0^2}{4}, \qquad \rho_{\rm gi0}^2 = \frac{c_0^2L^2}{2\omega_0^2r_0^2}, \qquad R_{\rm gi1}^2 = \frac{c_0^2L^2}{2\omega_0^2\rho_0^2}+\frac{r_0^2}{4}$$

- Loss of resolution due to the partial coherence of the source: $\rho_{\rm gi1} > \rho_{\rm gi0}$.
- Fully incoherent case $\rho_0 \to 0$: cf previous case $\rho_{\rm gi1} = \rho_{\rm gi0}$.
- Fully coherent case $r_0 = \rho_0$: the kernel is $H(\boldsymbol{x},\boldsymbol{r}) = \frac{\rho_0^4c_0^2}{64\omega_0^4\rho_{\rm gi1}^2}\exp\Big(-\frac{|\boldsymbol{x}|^2}{2\rho_{\rm gi1}^2}-\frac{|\boldsymbol{r}|^2}{2\rho_{\rm gi1}^2}\Big)$, which means there is no resolution at all.

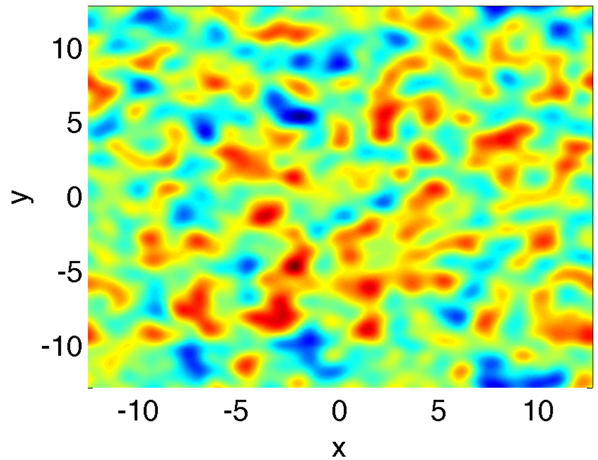
Wave propagation in a random medium

- Random medium model:

$$\frac{1}{c^2(\vec{\boldsymbol{x}})} = \frac{1}{c_0^2}\big(1+\mu(\vec{\boldsymbol{x}})\big)$$

c_0 is a reference speed,

$\mu(\vec{\boldsymbol{x}})$ is a zero-mean random process.



- The background Green's function (deterministic and known):

$$\Delta_{\vec{\boldsymbol{x}}}\hat{G}_0(\omega,\vec{\boldsymbol{x}},\vec{\boldsymbol{y}}) + \frac{\omega^2}{c_0^2}\hat{G}_0(\omega,\vec{\boldsymbol{x}},\vec{\boldsymbol{y}}) = -\delta(\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}})$$

The physical Green's function (random and unknown):

$$\Delta_{\vec{\boldsymbol{x}}}\hat{G}(\omega,\vec{\boldsymbol{x}},\vec{\boldsymbol{y}}) + \frac{\omega^2}{c_0^2}\big(1+\mu(\vec{\boldsymbol{x}})\big)\hat{G}(\omega,\vec{\boldsymbol{x}},\vec{\boldsymbol{y}}) = -\delta(\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}})$$

- A detailed stochastic analysis is possible in different regimes of separation of scales (small wavelength, large propagation distance, large bandwidth, ...).

Wave propagation in a random medium: the paraxial regime

- Consider the time-harmonic form of the scalar wave equation ($\vec{\boldsymbol{x}} = (\boldsymbol{x}, z)$)

$$(\partial_z^2 + \Delta_\perp)\hat{u} + \frac{\omega^2}{c_0^2}\big(1 + \mu(\boldsymbol{x}, z)\big)\hat{u} = 0.$$

Consider the paraxial regime $\lambda \ll l_c \ll L$. More precisely, in the scaled regime

$$\omega \to \frac{\omega}{\varepsilon^4}, \qquad \mu(\boldsymbol{x}, z) \to \varepsilon^3 \mu\big(\frac{\boldsymbol{x}}{\varepsilon^2}, \frac{z}{\varepsilon^2}\big),$$

the function $\hat{\phi}^\varepsilon$ defined by

$$\hat{u}^\varepsilon(\omega, \boldsymbol{x}, z) = e^{i\frac{\omega z}{\varepsilon^4 c_0}} \hat{\phi}^\varepsilon\big(\frac{\omega}{\varepsilon^4}, \frac{\boldsymbol{x}}{\varepsilon^2}, z\big)$$

satisfies

$$\varepsilon^4 \partial_z^2 \hat{\phi}^\varepsilon + \left(2i\frac{\omega}{c_0}\partial_z \hat{\phi}^\varepsilon + \Delta_\perp \hat{\phi}^\varepsilon + \frac{\omega^2}{c_0^2}\frac{1}{\varepsilon}\mu\big(\boldsymbol{x}, \frac{z}{\varepsilon^2}\big)\hat{\phi}^\varepsilon\right) = 0.$$

[1] J. Garnier and K. Sølna, Ann. Appl. Probab. **19** 318 (2009).

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satisfies

$$\varepsilon^4 \partial_z^2 \hat{\phi}^\varepsilon + \Big(2i\frac{\omega}{c_0}\partial_z \hat{\phi}^\varepsilon + \Delta_\perp \hat{\phi}^\varepsilon + \frac{\omega^2}{c_0^2}\frac{1}{\varepsilon}\mu\big(\boldsymbol{x}, \frac{z}{\varepsilon^2}\big)\hat{\phi}^\varepsilon\Big) = 0.$$

- In the regime $\varepsilon \ll 1$, the forward-scattering approximation in direction z is valid and $\hat{\phi} = \lim_{\varepsilon \to 0} \hat{\phi}^\varepsilon$ satisfies the Itô-Schrödinger equation [1]

$$2i\frac{\omega}{c_0}\partial_z \hat{\phi} + \Delta_\perp \hat{\phi} + \frac{\omega^2}{c_0^2}\dot{B}(\boldsymbol{x}, z)\hat{\phi} = 0$$

with $\dot{B}(\boldsymbol{x}, z)$ white noise $\mathbb{E}[B(\boldsymbol{x}, z)B(\boldsymbol{x}', z')] = \gamma(\boldsymbol{x} - \boldsymbol{x}')\,\delta(z - z')$,
$\gamma(\boldsymbol{x}) = \int_{-\infty}^{\infty} \mathbb{E}[\mu(\boldsymbol{0}, 0)\mu(\boldsymbol{x}, z)]dz$.

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satisfies

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- In the regime $\varepsilon \ll 1$, the forward-scattering approximation in direction z is valid and $\hat{\phi} = \lim_{\varepsilon \to 0} \hat{\phi}^\varepsilon$ satisfies the Itô-Schrödinger equation [1]

$$d\hat{\phi} = \frac{ic_0}{2\omega}\Delta_\perp \hat{\phi} dz + \frac{i\omega}{2c_0}\hat{\phi} \circ dB(\boldsymbol{x}, z)$$

with $B(\boldsymbol{x}, z)$ Brownian field $\mathbb{E}[B(\boldsymbol{x}, z)B(\boldsymbol{x}', z')] = \gamma(\boldsymbol{x} - \boldsymbol{x}')\, z \wedge z'$, $\gamma(\boldsymbol{x}) = \int_{-\infty}^{\infty} \mathbb{E}[\mu(\boldsymbol{0}, 0)\mu(\boldsymbol{x}, z)]dz$.

[1] J. Garnier and K. Sølna, Ann. Appl. Probab. **19** 318 (2009).

Moment calculations in the random paraxial regime

Consider

$$d\hat{\phi} = \frac{ic_0}{2\omega}\Delta_\perp \hat{\phi} dz + \frac{i\omega}{2c_0}\hat{\phi} \circ dB(\boldsymbol{x}, z)$$

starting from $\hat{\phi}(\boldsymbol{x}, z=0) = f(\boldsymbol{x})$.

- By Itô's formula,

$$\frac{d}{dz}\mathbb{E}[\hat{\phi}] = \frac{ic_0}{2\omega}\Delta_\perp \mathbb{E}[\hat{\phi}] - \frac{\omega^2 \gamma(\mathbf{0})}{8c_0^2}\mathbb{E}[\hat{\phi}]$$

and therefore

$$\mathbb{E}\big[\hat{\phi}(\boldsymbol{x}, z)\big] = \hat{\phi}_0(\boldsymbol{x}, z) \exp\Big(- \frac{\gamma(\mathbf{0})\omega^2 z}{8c_0^2}\Big),$$

where $\gamma(\boldsymbol{x}) = \int_{-\infty}^{\infty} \mathbb{E}[\mu(\mathbf{0}, 0)\mu(\boldsymbol{x}, z)]dz$ and $\hat{\phi}_0$ is the solution in the homogeneous medium.

$\hookrightarrow$ Strong damping of the coherent wave, with the scattering mean free path $l_{\rm sca} = 8c_0^2/(\gamma(\mathbf{0})\omega^2)$.

• The mean Wigner transform defined by

$$W(z,\boldsymbol{x},\boldsymbol{q}) = \int \exp\big(-i\boldsymbol{q}\cdot\boldsymbol{y}\big)\mathbb{E}\Big[\hat{\phi}\big(z,\boldsymbol{x}+\frac{\boldsymbol{y}}{2}\big)\overline{\hat{\phi}}\big(z,\boldsymbol{x}-\frac{\boldsymbol{y}}{2}\big)\Big]\, d\boldsymbol{y},$$

is the angularly-resolved mean wave energy density. By Itô's formula,

$$\frac{\partial W}{\partial z} + \frac{c_0}{\omega}\boldsymbol{q}\cdot\nabla_{\boldsymbol{x}} W = \frac{\omega^2}{4(2\pi)^2 c_0^2}\int \hat{\gamma}(\boldsymbol{k})\Big[W(\boldsymbol{q}-\boldsymbol{k}) - W(\boldsymbol{q})\Big] d\boldsymbol{k},$$

starting from $W(z=0,\boldsymbol{x},\boldsymbol{q}) = W_0(\boldsymbol{x},\boldsymbol{q})$, the Wigner transform of the initial field f.

$$\begin{aligned} W(z,\boldsymbol{x},\boldsymbol{q}) &= \frac{1}{(2\pi)^2}\iint \exp\Big(i\boldsymbol{\xi}\cdot\big(\boldsymbol{x}-\boldsymbol{q}\frac{c_0 z}{\omega}\big) - i\boldsymbol{y}'\cdot\boldsymbol{q}\Big)\hat{W}_0\big(\boldsymbol{\xi},\boldsymbol{y}'\big) \\ &\quad \times \exp\Big(\frac{\omega^2}{4c_0^2}\int_0^z \gamma\big(\boldsymbol{y}'+\boldsymbol{\xi}\frac{c_0 z'}{\omega}\big) - \gamma(\boldsymbol{0})dz'\Big) d\boldsymbol{\xi} d\boldsymbol{y}', \end{aligned}$$

where $\hat{W}_0$ is defined in terms of the initial field f as:

$$\hat{W}_0(\boldsymbol{\xi},\boldsymbol{y}) = \int \exp\big(-i\boldsymbol{\xi}\cdot\boldsymbol{x}\big) f\big(\boldsymbol{x}+\frac{\boldsymbol{y}}{2}\big)\overline{f}\big(\boldsymbol{x}-\frac{\boldsymbol{y}}{2}\big) d\boldsymbol{x}.$$

• When $L \gg l_{\mathrm{sca}}$: diffusion in $\boldsymbol{q}$-space; one can identify the transport mean free path $l_{\mathrm{tr}} = l_{\mathrm{sca}} l_{\mathrm{cor}}^2 \omega_0^2 / c_0^2$. It is $\gg l_{\mathrm{sca}}$ (strongly forward scattering).
Here the correlation radius of the medium l_{cor} is defined by:

$$\gamma(\boldsymbol{x}) = \gamma(\boldsymbol{0})[1 - |\boldsymbol{x}|^2/l_{\mathrm{cor}}^2 + o(|\boldsymbol{x}|^2/l_{\mathrm{cor}}^2)].$$

- In a random medium, by Itô's formula, one can write a closed-form equation for the n-th order moment.

Depending on the regime ($l_c \ll r_0$ or $l_c \gg r_0$, where r_0 is the beam width), the wave fluctuations may have Gaussian statistics (scintillation regime) or not (spot-dancing regime) [1].

[1] J. Garnier and K. Sølna, *Comm. Part. Differ. Equat.* **39**, 626 (2014), preprint (2015).

- Let us consider the fourth-order moment:

$$\begin{aligned} M_4(z,\boldsymbol{r}_1,\boldsymbol{r}_2,\boldsymbol{q}_1,\boldsymbol{q}_2) &= \mathbb{E}\Big[\hat{\phi}\big(z,\frac{\boldsymbol{r}_1+\boldsymbol{r}_2+\boldsymbol{q}_1+\boldsymbol{q}_2}{2}\big)\hat{\phi}\big(z,\frac{\boldsymbol{r}_1-\boldsymbol{r}_2+\boldsymbol{q}_1-\boldsymbol{q}_2}{2}\big) \\ &\quad \times\overline{\hat{\phi}}\big(z,\frac{\boldsymbol{r}_1+\boldsymbol{r}_2-\boldsymbol{q}_1-\boldsymbol{q}_2}{2}\big)\overline{\hat{\phi}}\big(z,\frac{\boldsymbol{r}_1-\boldsymbol{r}_2-\boldsymbol{q}_1+\boldsymbol{q}_2}{2}\big)\Big] \end{aligned}$$

Take Fourier transform

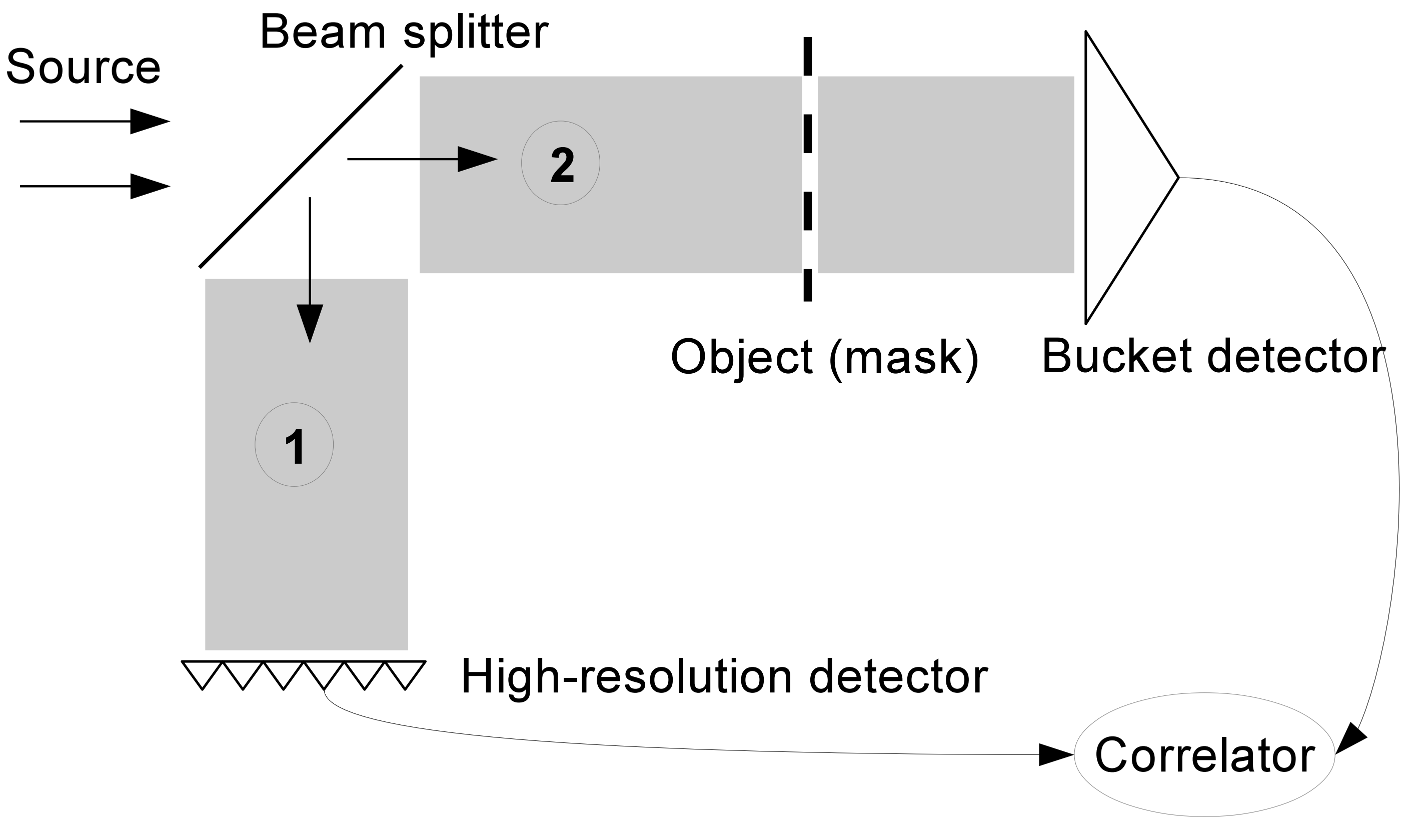
$$\begin{aligned} \hat{M}_4(z,\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2) &= \iint M_4(z,\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{r}_1,\boldsymbol{r}_2) \\ &\quad \times \exp\big(-i\boldsymbol{q}_1\cdot\boldsymbol{\xi}_1-i\boldsymbol{r}_1\cdot\boldsymbol{\zeta}_1-i\boldsymbol{q}_2\cdot\boldsymbol{\xi}_2-i\boldsymbol{r}_2\cdot\boldsymbol{\zeta}_2\big)d\boldsymbol{r}_1 d\boldsymbol{r}_2 d\boldsymbol{q}_1 d\boldsymbol{q}_2 \end{aligned}$$

$\hat{M}_4$ satisfies

$$\begin{aligned} \frac{\partial\hat{M}_4}{\partial z}+\frac{ic_0}{\omega}(\boldsymbol{\xi}_1\cdot\boldsymbol{\zeta}_1+\boldsymbol{\xi}_2\cdot\boldsymbol{\zeta}_2)\hat{M}_4 &= \frac{\omega^2}{4(2\pi)^2c_0^2}\int\hat{\gamma}(\boldsymbol{k})\Big[\hat{M}_4(\boldsymbol{\xi}_1-\boldsymbol{k},\boldsymbol{\xi}_2-\boldsymbol{k},\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2) \\ &+\hat{M}_4(\boldsymbol{\xi}_1-\boldsymbol{k},\boldsymbol{\xi}_2,\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2-\boldsymbol{k})+\hat{M}_4(\boldsymbol{\xi}_1+\boldsymbol{k},\boldsymbol{\xi}_2-\boldsymbol{k},\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2) \\ &+\hat{M}_4(\boldsymbol{\xi}_1+\boldsymbol{k},\boldsymbol{\xi}_2,\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2-\boldsymbol{k})-2\hat{M}_4(\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2) \\ &-\hat{M}_4(\boldsymbol{\xi}_1,\boldsymbol{\xi}_2-\boldsymbol{k},\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2-\boldsymbol{k})-\hat{M}_4(\boldsymbol{\xi}_1,\boldsymbol{\xi}_2+\boldsymbol{k},\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2-\boldsymbol{k})\Big]d\boldsymbol{k} \end{aligned}$$

[1] J. Garnier and K. Sølna, *Comm. Part. Differ. Equat.* **39**, 626 (2014), preprint (2015).

Ghost imaging in heterogeneous media



The medium in paths 1 and 2 is heterogeneous (for instance, turbulent atmosphere). They are two independent realizations of the same distribution.

- If the correlation function of the noise sources is

$$\Big\langle n(t,\boldsymbol{x})\overline{n(t',\boldsymbol{x}')}\Big\rangle = F(t-t')\exp\Big(-\frac{|\boldsymbol{x}|^2}{r_0^2}\Big)\delta(\boldsymbol{x}-\boldsymbol{x}'),$$

if we denote the integrated correlation function of the medium fluctuations by

$$\gamma(\boldsymbol{x}) = \int_{-\infty}^{\infty} \mathbb{E}[\mu(\mathbf{0},0)\mu(\boldsymbol{x},z)]dz$$

- then

$$C^{(1)}(\boldsymbol{x}) = \int_{\mathbb{R}^2} \mathcal{H}(\boldsymbol{x}-\boldsymbol{y})|\mathcal{T}(\boldsymbol{y})|^2 d\boldsymbol{y},$$

with

$$\mathcal{H}(\boldsymbol{x}) = \frac{r_0^4}{2^9\pi^3L^4}\int_{\mathbb{R}^2} d\boldsymbol{\beta}\exp\Big(-\frac{|\boldsymbol{\beta}|^2}{2} - \frac{\gamma_2(r_0\boldsymbol{\beta})\omega_0^2L}{2c_0^2} + i\frac{\omega_0 r_0\boldsymbol{x}\cdot\boldsymbol{\beta}}{c_0L}\Big).$$

and $\gamma_2(\boldsymbol{x}) = \int_0^1 \gamma(\mathbf{0}) - \gamma(\boldsymbol{x}s)ds$.

• If the medium is strongly scattering, in the sense that the propagation distance is larger than the scattering mean free path $L/l_{\rm sca} \gg 1$, with

$$l_{\rm sca} = \frac{8c_0^2}{\gamma(\mathbf{0})\omega_0^2},$$

• then

$$\mathcal{H}(\boldsymbol{x}) = \frac{r_0^4 \rho_{\rm gi0}^2}{2^8 \pi^2 L^4 \rho_{\rm gi2}^2} \exp\Big(-\frac{|\boldsymbol{x}|^2}{4\rho_{\rm gi2}^2}\Big),$$

with

$$\rho_{\rm gi2}^2 = \rho_{\rm gi0}^2 + \frac{4c_0^2 L^3}{3\omega_0^2 l_{\rm sca} l_{\rm cor}^2}, \qquad \rho_{\rm gi0}^2 = \frac{c_0^2 L^2}{2\omega_0^2 r_0^2}$$

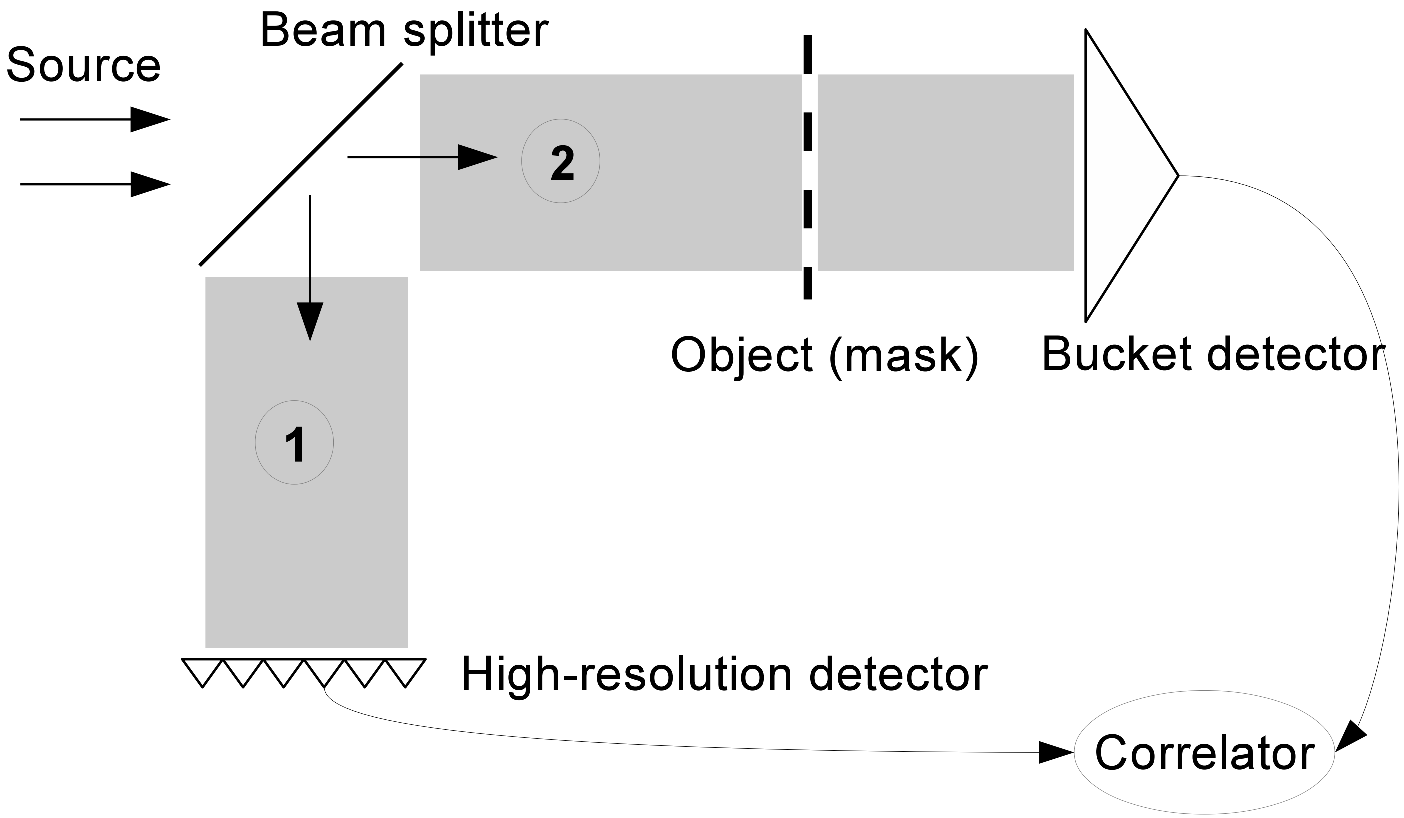
and the correlation radius of the medium $l_{\rm cor}$ is defined by:

$$\gamma(\boldsymbol{x}) = \gamma(\mathbf{0})[1 - |\boldsymbol{x}|^2/l_{\rm cor}^2 + o(|\boldsymbol{x}|^2/l_{\rm cor}^2)].$$

$\hookrightarrow$ Scattering only slightly reduces the resolution !

This imaging method is robust with respect to medium noise. It gives an image even when $L/l_{\rm sca} \gg 1$. The image resolution is reduced when $L/l_{\rm tr} > 1$.

Ghost imaging in heterogeneous identical media



The medium in paths 1 and 2 is heterogeneous (for instance, turbulent atmosphere). Imagine that they are the *same realization* (just for the beauty of the analysis).

• If the correlation function of the noise sources is

$$\left\langle n(t,\boldsymbol{x})\overline{n(t',\boldsymbol{x}')}\right\rangle = F(t-t')\exp\Big(-\frac{|\boldsymbol{x}|^2}{r_0^2}\Big)\delta(\boldsymbol{x}-\boldsymbol{x}'),$$

if we denote the integrated correlation function of the medium fluctuations by

$$\gamma(\boldsymbol{x}) = \int_{-\infty}^{\infty} \mathbb{E}[\mu(\mathbf{0},0)\mu(\boldsymbol{x},z)]dz,$$

if $L/l_{\rm sca} \gg 1$, with

$$l_{\rm sca} = \frac{8c_0^2}{\gamma(\mathbf{0})\omega_0^2},$$

if $\gamma(\boldsymbol{x}) = \gamma(\mathbf{0})[1-|\boldsymbol{x}|^2/l_{\rm cor}^2 + o(|\boldsymbol{x}|^2/l_{\rm cor}^2)]$,

• then

$$C^{(1)}(\boldsymbol{x}) = \int_{\mathbb{R}^2} \mathcal{H}(\boldsymbol{x}-\boldsymbol{y})|\mathcal{T}(\boldsymbol{y})|^2 d\boldsymbol{y},$$

with

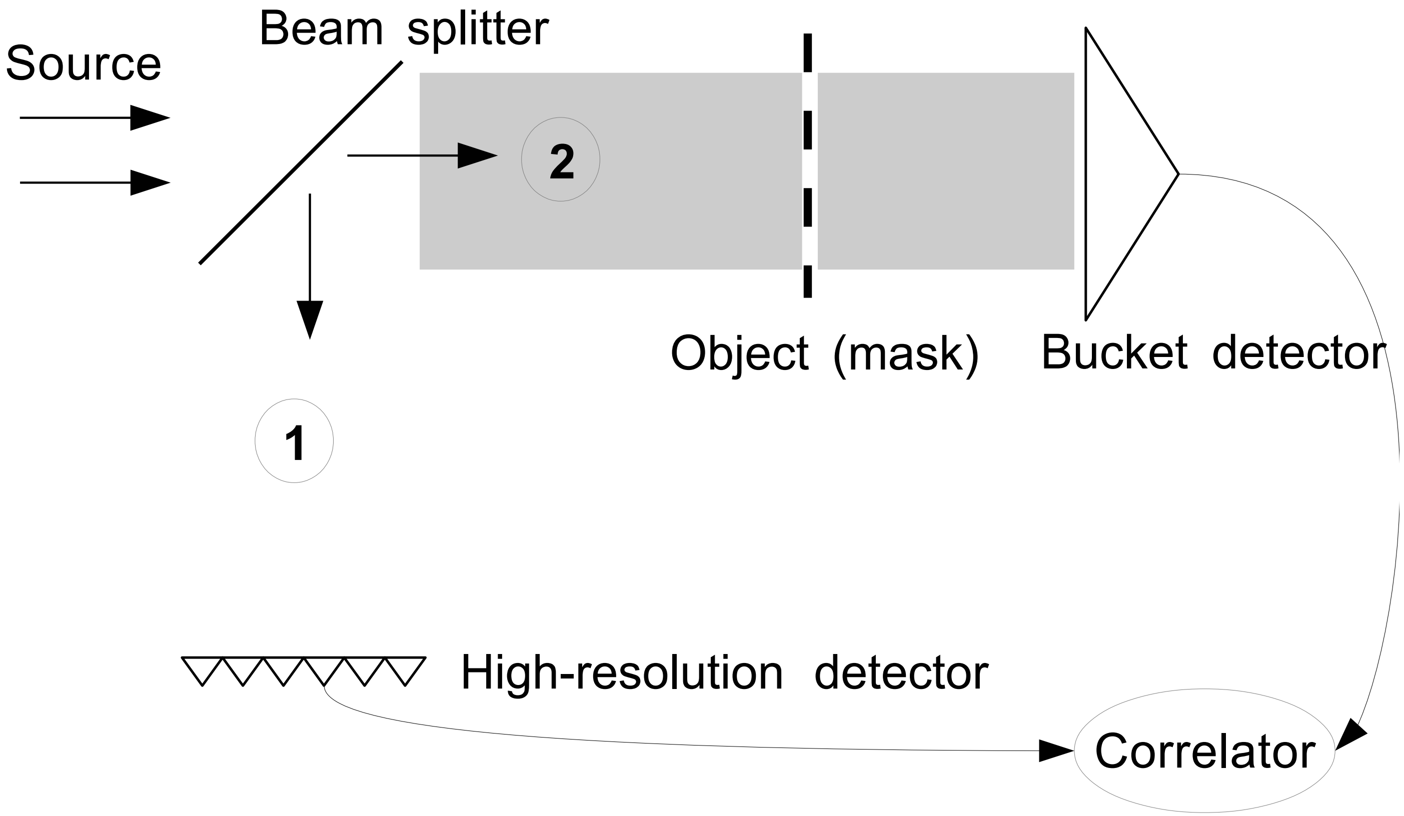
$$\mathcal{H}(\boldsymbol{x}) = \frac{r_0^4}{2^8\pi^2L^4}\exp\Big(-\frac{|\boldsymbol{x}|^2}{4\rho_{\rm gi3}^2}\Big),$$

with the radius

$$\frac{1}{\rho_{\rm gi3}^2} = \frac{1}{\rho_{\rm gi0}^2} + \frac{16L}{l_{\rm sca}l_{\rm cor}^2}$$

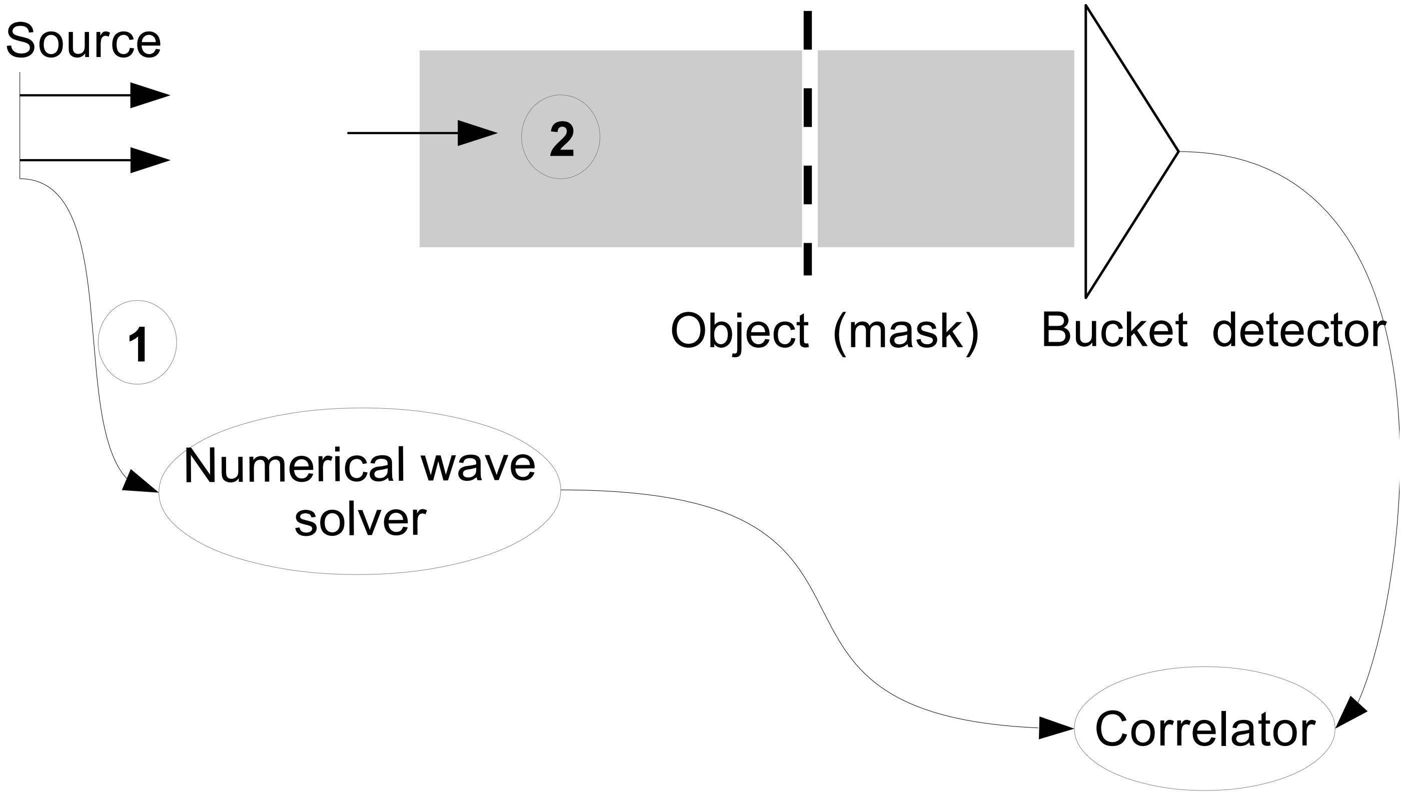
↪ the radius of the convolution kernel is reduced by scattering and can even be smaller than the Rayleigh resolution formula: enhanced resolution compared to the homogeneous case (similar phenomenon observed in time-reversal experiments) !

Ghost imaging in heterogeneous/homogeneous media



The medium in path 2 is heterogeneous. The medium in path 1 is homogeneous.

Ghost imaging with a virtual high-resolution detector



- The medium in path 2 is randomly heterogeneous.

- There is no other measurement than the integrated transmitted intensity $I_2(t)$.

- The realization of the source is known (use of a Spatial Light Modulator) and the medium is taken to be homogeneous in the "virtual path 1" $\rightarrow$ one can *compute* the field (and therefore its intensity $I_1(t, \boldsymbol{x})$) in the "virtual" output plane of path 1 [J.H. Shapiro, Phys. Rev. A, 78 (2008)].

• If the correlation function of the noise sources is

$$\left\langle n(t,\boldsymbol{x})\overline{n(t',\boldsymbol{x}')}\right\rangle = F(t-t')\exp\Big(-\frac{|\boldsymbol{x}|^2}{r_0^2}\Big)\delta(\boldsymbol{x}-\boldsymbol{x}'),$$

if we denote the integrated correlation function of the medium fluctuations by

$$\gamma(\boldsymbol{x}) = \int_{-\infty}^{\infty} \mathbb{E}[\mu(\mathbf{0},0)\mu(\boldsymbol{x},z)]dz,$$

if $L/l_{\rm sca} \gg 1$, with

$$l_{\rm sca} = \frac{8c_0^2}{\gamma(\mathbf{0})\omega_0^2},$$

if $\gamma(\boldsymbol{x}) = \gamma(\mathbf{0})[1-|\boldsymbol{x}|^2/l_{\rm cor}^2 + o(|\boldsymbol{x}|^2/l_{\rm cor}^2)]$,

• then

$$C^{(1)}(\boldsymbol{x}) = \int_{\mathbb{R}^2} \mathcal{H}(\boldsymbol{x}-\boldsymbol{y})|\mathcal{T}(\boldsymbol{y})|^2 d\boldsymbol{y},$$

with

$$\mathcal{H}(\boldsymbol{x}) = \frac{r_0^4\rho_{\rm gi0}^2}{2^8\pi^2L^4\rho_{\rm gi4}^2}\exp\Big(-\frac{|\boldsymbol{x}|^2}{4\rho_{\rm gi4}^2}\Big),$$

with the radius

$$\rho_{\rm gi4}^2 = \rho_{\rm gi0}^2 + \frac{2c_0^2L^3}{3\omega_0^2 l_{\rm sca} l_{\rm cor}^2}.$$

↪ a *one-pixel camera* can give a high-resolution image of the object in scattering media!

Conclusion

- Imaging with noise sources and/or through scattering media is possible using correlation-based techniques.
- First application (with time-resolved measurements): seismic interferometry.
- Second application (with intensity only measurements): ghost imaging.
- Many other applications !