Relationships Between Interferometric Theories

+Near-Surface Green's Functions

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Blackboard Notes from 5/15/2015 (Alternative to 1st half of PPT presentation)

1. Relationships Between Interferometric Theories
2. Simplified Surface-Wave Inversion

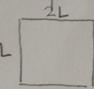
5/14/15
VCT

Assumptions: ① Equipartition of Modes
② Isotropic Illumination
③ Homogeneous Source Distribution

- Relationship
- Not time to discuss 'realistic' anisotropic noise sources, but see talk by K. Wapenaar
- Intuitive understanding with analog example.

Example is scalar waves on a periodic square.

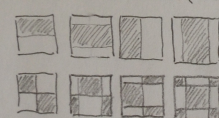
i.e. $\frac{\partial^2 u}{\partial t^2} = c(x,y)^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$



- Periodic boundary conditions
- Very simplified but has many characteristics of Earth

Modal solution can be intuited (when $c(x,y)=c$)

Pictorially:



'2 node modes' — all of these 4 modes have frequency $\omega_k^2 = \frac{\pi^2 c^2 \cdot 1}{L^2} \leftarrow 1^2+0^2$

'4 node modes' (1 in x, 1 in y) — these modes have freq. $\omega_k^2 = \frac{\pi^2 c^2 \cdot 2}{L^2} \leftarrow 1^2+1^2$

Generally, $u(\vec{x},t) = \sum A_k \cos\frac{m\pi x}{L} \cos\frac{n\pi y}{L} \cos(\omega_k t + \phi_{k1})$ where $\omega_k^2 = \frac{\pi^2 c^2 (m^2+n^2)}{L^2}$

$+ B_k \cos \cdot \sin \cdot \cos$

$+ C_k \sin \cdot \cos \cdot \cos$

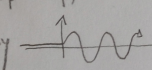
$+ D_k \sin \cdot \sin \cdot \cos$

$= \sum \alpha_p s_p(\vec{x}) \cos(\omega_p t + \phi_p)$

$\left(\frac{\#\text{nodes in } x}{2}, \frac{\#\text{nodes in } y}{2}\right)$

$(1,7), (5,5), (7,1) \rightarrow$ same freq 12 modes

Green's Functions for modes are easy



& excitation proportional to mode shape at force location

$G(\vec{x},\vec{x}_0,t) = \sum \frac{s_p(\vec{x}) s_p(\vec{x}_0)}{\omega_p} \cdot \sin\omega_p t, \; t>0$

$= 0, \; t<0$

$\rightarrow G^{Ex} = G(t) - G(-t) = \sum \frac{s_p(\vec{x}) s_p(\vec{x}_0)}{\omega_p} \sin\omega_p t$

Correlations for modes (sin waves) are easy

$C(\underbrace{\cos(\omega_1 t+\phi_1)}_{f}, \underbrace{\cos(\omega_2 t+\phi_2)}_{g}) = \begin{cases} 0 & \text{if } \omega_1 \neq \omega_2 \\ \frac{1}{2}\cos(\omega_1 t + (\phi_2-\phi_1)) & \text{if } \omega_1 = \omega_2 \end{cases}$

Note: Modes of diff. freq. automatically decouple in correlation (even without 'noise')

For $\omega_1=\omega_2$, if phases take on slowly changing, uncorrelated values then $C(f,g)=0$ & only $C(f,f)=\frac{1}{2}\cos\omega_1 t$

then $C(u(\vec{x}_1),u(\vec{x}_2)) = \frac{1}{2}\sum \alpha_p^2 s_p(\vec{x}_1) s_p(\vec{x}_2) \cos\omega_p t$. Same form as G^{Ex}! (except sin vs. cos)

Note: α_p do not need to satisfy equipartition for a relationship to hold! e.g. $\alpha_k=\alpha$ (mode amps equal, not energy)

then $C(u(\vec{x}_1),u(\vec{x}_2)) = \frac{1}{2}\alpha^2 \frac{dG^{Ex}}{dt}$ c.f. $\frac{dC}{dt} = -\frac{1}{2}\alpha^2 G^{Ex}$ for equipartition. Note same phase relationship.

① Standing → traveling waves:
Equipartition within each freq implies "discretely isotropic", i.e. isotropic in all allowable directions
For high freq → isotropic if medium has <u>symmetry</u> (e.g. physics is invariant to changes in azimuth) if $c(x,y) \neq c$ not isotropic

② Isotropic (with symmetry) implies equipartition within each freq. family (Note: higher order & fundamental <u>not</u> necessarily equipartitioned)

③ Homogeneous forcing & symmetry implies equipartion within each family.

3 Different Noise Correlation Assumptions

Equipartitioned Modes	Isotropic Noise Field	Homogeneous Noise Sources
Lobkis & Weaver 2001	Snieder 2004	Wapenaar 2004

$$\frac{dC_{xy}}{dt} = -A\left[G_{xy}(t) - G_{xy}(-t)\right]$$

- What is the relationship between these derivations?
- Why are results similar despite different assumptions?
- Under realistic conditions, what's still approx. true?

Mode Derivation

- Wave equation: $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2} = L[u]$

- Modes: $L[s_k] = \lambda_k s_k = -\omega_k^2 s_k$

- General solution: $u(\mathbf{x},t) = \sum_k a_k s_k(\mathbf{x}) \cos(\omega_k t + \phi_k)$

- Extended Green's function:

$$G^{Ex}(\mathbf{x},t;\mathbf{x}_0) = G(\mathbf{x},t;\mathbf{x}_0) - G(\mathbf{x},-t;\mathbf{x}_0)$$

$$= \sum \frac{1}{\omega_k} s_k(\mathbf{x}) s_k(\mathbf{x}_0) \sin \omega_k t$$

Mode Derivation

- Correlations: $f(t) = \cos(\omega_k t + \phi_k)$ & $g(t) = \cos(\omega_{k'} t + \phi_{k'})$

$$C_{fg}(t) = \begin{cases} \frac{1}{2}\cos(\omega_k t + \phi_{k'} - \phi_k) & \text{if } \omega_k = \omega_{k'} \\ 0 & \text{if } \omega_k \neq \omega_{k'} \end{cases}$$

- If energy equipartitioned: $u(\mathbf{x},t) = A\sum_k \frac{s_k(\mathbf{x})}{\omega_k}\cos(\omega_k t + \phi_k)$

- If mode amplitudes equal: $u(\mathbf{x},t) = A\sum_k s_k(\mathbf{x})\cos(\omega_k t + \phi_k)$

Mode Derivation

- If energy equipartitioned: $u(\mathbf{x},t) = A\sum_k \frac{s_k(\mathbf{x})}{\omega_k}\cos(\omega_k t + \phi_k)$

$$C_{x1x2}(t) = \frac{A^2}{2}\sum \frac{s_k(x_1)s_k(x_2)}{\omega_k^2}\cos\omega_k t$$

$$\rightarrow \quad \boxed{-\frac{dC_{x1x2}(t)}{dt} = \frac{A^2}{2}G^{Ex}}$$

- If mode amplitudes equal: $u(\mathbf{x},t) = A\sum_k s_k(\mathbf{x})\cos(\omega_k t + \phi_k)$

$$C_{x1x2}(t) = \frac{A^2}{2}\sum s_k(x_1)s_k(x_2)\cos\omega_k t$$

$$\rightarrow \quad \boxed{C_{x1x2}(t) = \frac{A^2}{2}\frac{dG^{Ex}}{dt}}$$

True for deterministic problem
Requires $\omega_k \neq \omega_{k'}$ Note spectrum

Is modal equipartition equivalent to isotropic?

- Note: Standing modes can be written as traveling waves
- If there is spatial symmetry (e.g. physics is invariant to a change in azimuth), equipartition implies isotropic

- Velocity heterogeneity can cause problems
- Isotropic implies equipartition of a subset of modes

Tsai 2010

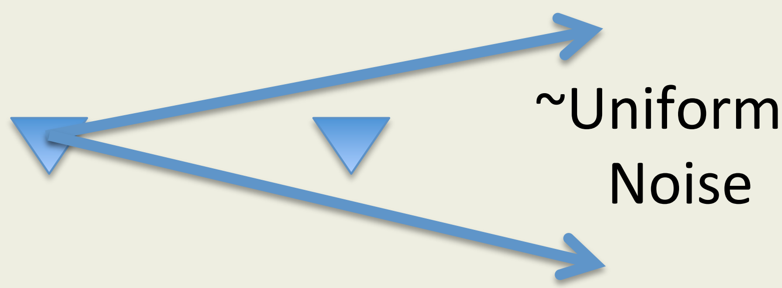
Is homogeneous excitation equivalent to equipartitioned?

- If modes have symmetries…
  - Yes, implies equipartition within each mode family
- Homogeneous 2D ≠ homogeneous 3D
  - e.g. modes with different depth sensitivities excited differently by surface sources, so homogeneous 2D does not result in equipartition

Tsai 2010

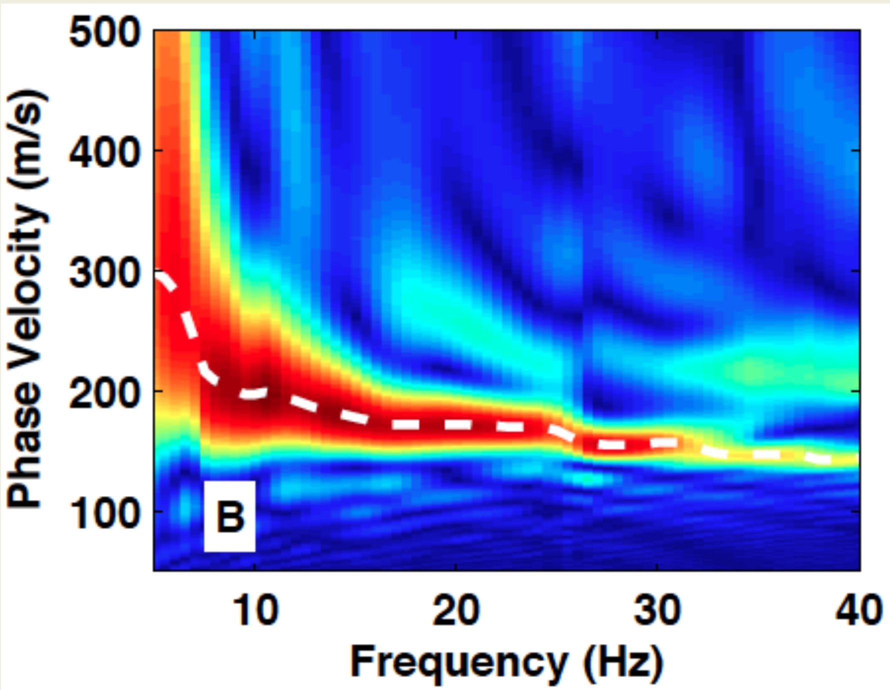
What if only subset of directions equipartitioned?

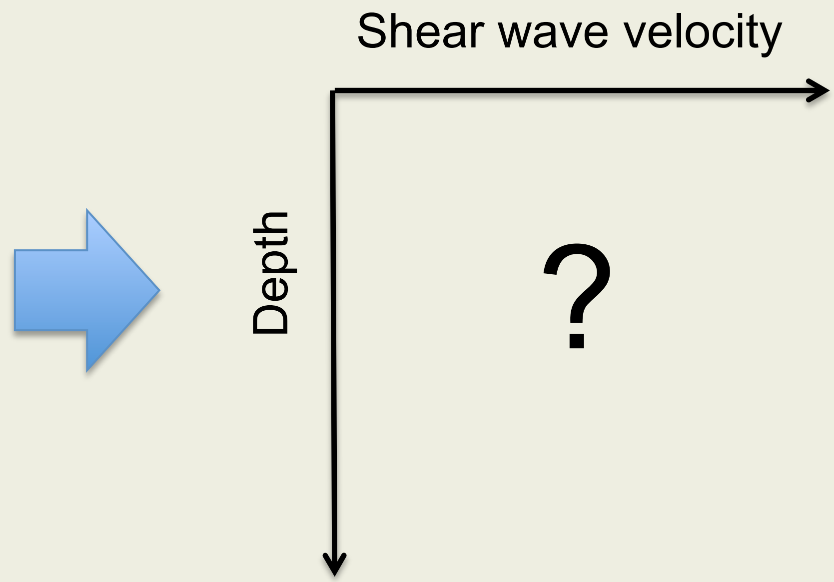
- If equipartition does not hold for a family of modes, the theory cannot be used
- But stationary phase arguments (e.g. Snieder 2004) still work for rule on how isotropic is necessary



Tsai 2010

Question: How to Invert Phase Velocities for Shear Wave Velocities at Depth

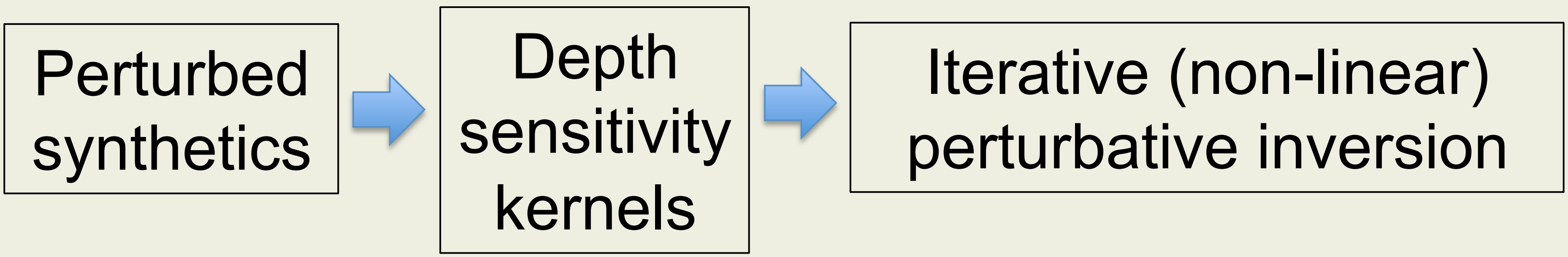




- Classical Approach (Perturbational)
- 'Dix-like' Approach (Linear, approx but fast & easy)

Classical Phase Velocity Inversion

- Begin with starting velocity model and synthetic phase velocities
- Perturbing model at range of depths and rerunning synthetics gives sensitivities for inversion



Difficulties:

- Need starting model
- Synthetics take 10’s of seconds (to minutes) to run
- Code is 1000’s of lines long (nontrivial, black-box)

Near-Surface Green's Functions

- Usefulness of exact results for approximate structure vs. approx. results for exact structure?
- Let's evaluate GF for power law vel.: $\beta \approx \beta_0 \left(\frac{z}{z_0} \right)^{\alpha}$

eigenfunctions

geometric decay

$$G(\omega, z, h) = \frac{l_1(z) l_1(h)}{cUI} e^{i(kr+\pi/4)} \sqrt{\frac{2}{\pi kr}} \cdot f(\theta)$$

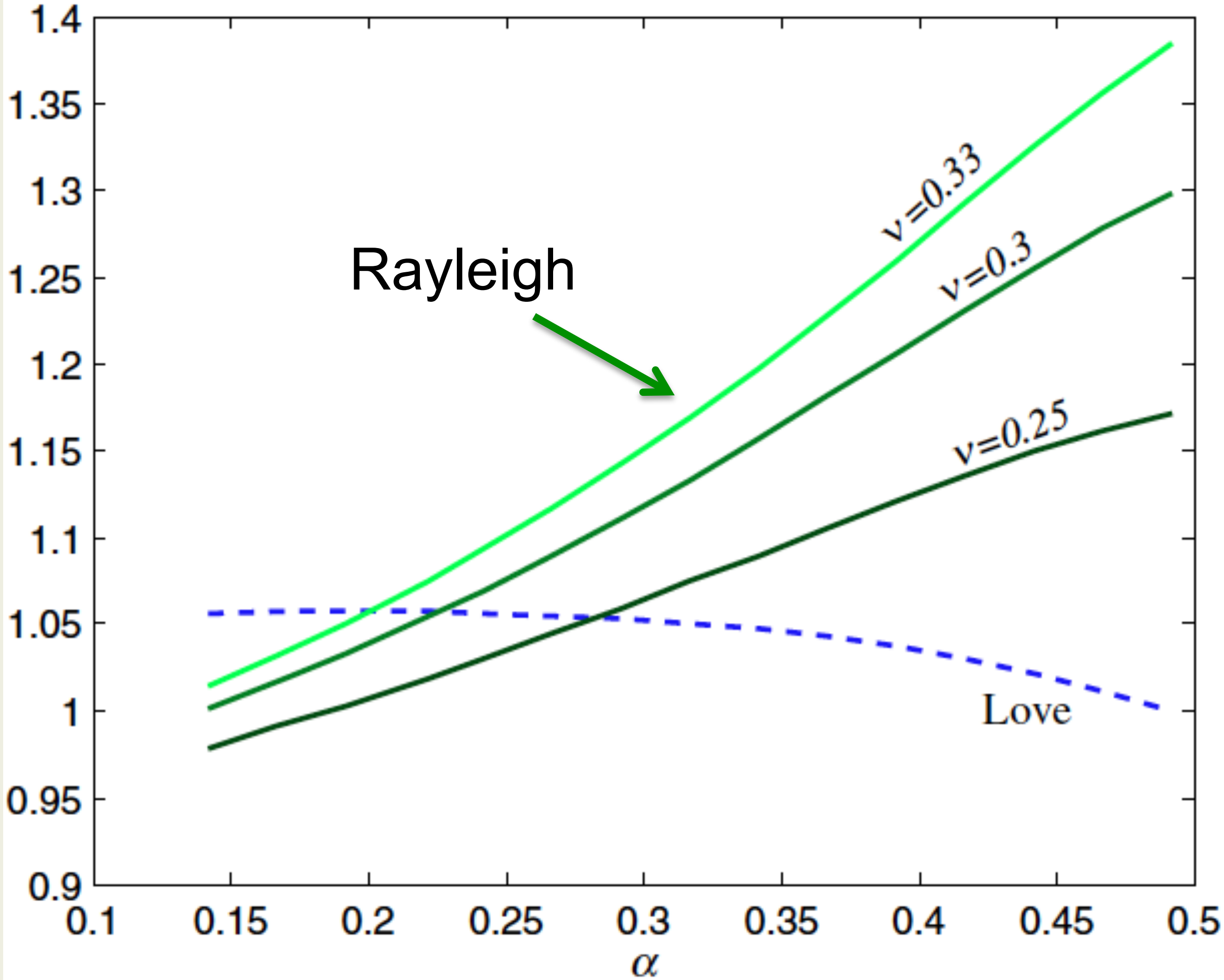
phase

azimuth dependence

- Need to determine eigenfunctions and c(ω)…
- c(ω) scaling is determined! $c(\omega) = \frac{\omega}{k} = \ldots = c_0 \left(\frac{\omega}{\omega_0} \right)^{-\frac{\alpha}{1-\alpha}}$
- Analytic result (2 consts.)

Phase Velocity Coeff. for Rayleigh and Love Waves

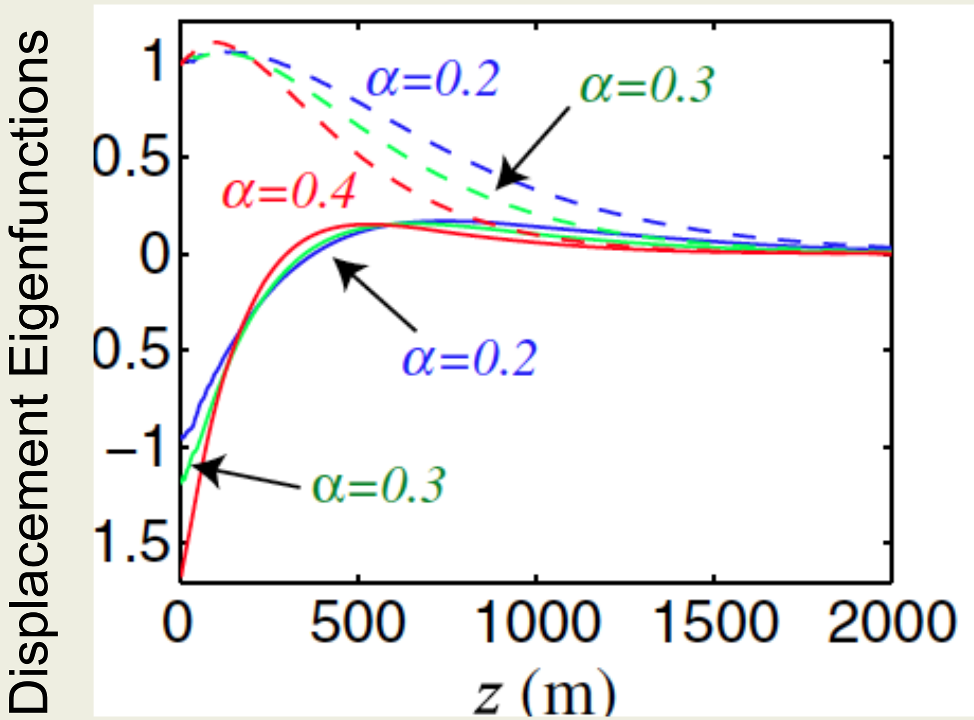
Scaled phase velocity vs. power-law exponent α



Tsai and Atiganyanun, BSSA 2014

An Alternative Phase Velocity Inversion

- Analogous to the Dix equation from reflection seismology
- Assume approximate analytic eigenfunctions
- Realistic Earth models generally easily approximated!
- Rayleigh's principle: ε changes to eigenfunctions do not change phase velocities!



All power-law velocity profile eigenfunctions can be approximated as

$$r_1 \sim c_1 e^{-a_1 k z} + c_2 e^{-a_2 k z}$$

Not a bad approx. for arbitrary realistic structure

An Alternative Phase Velocity Inversion

- With known eigenfunctions, phase velocities determined

$$\omega^2 I_1 - k^2 I_2 - k I_3 - I_4 = 0 \quad \Rightarrow \quad c(\omega) = \omega / k$$

- Classical approach would perturb ρ, λ, μ at each depth and iteratively adjust model until convergence
- With approx. eigenfunctions, inversion is direct

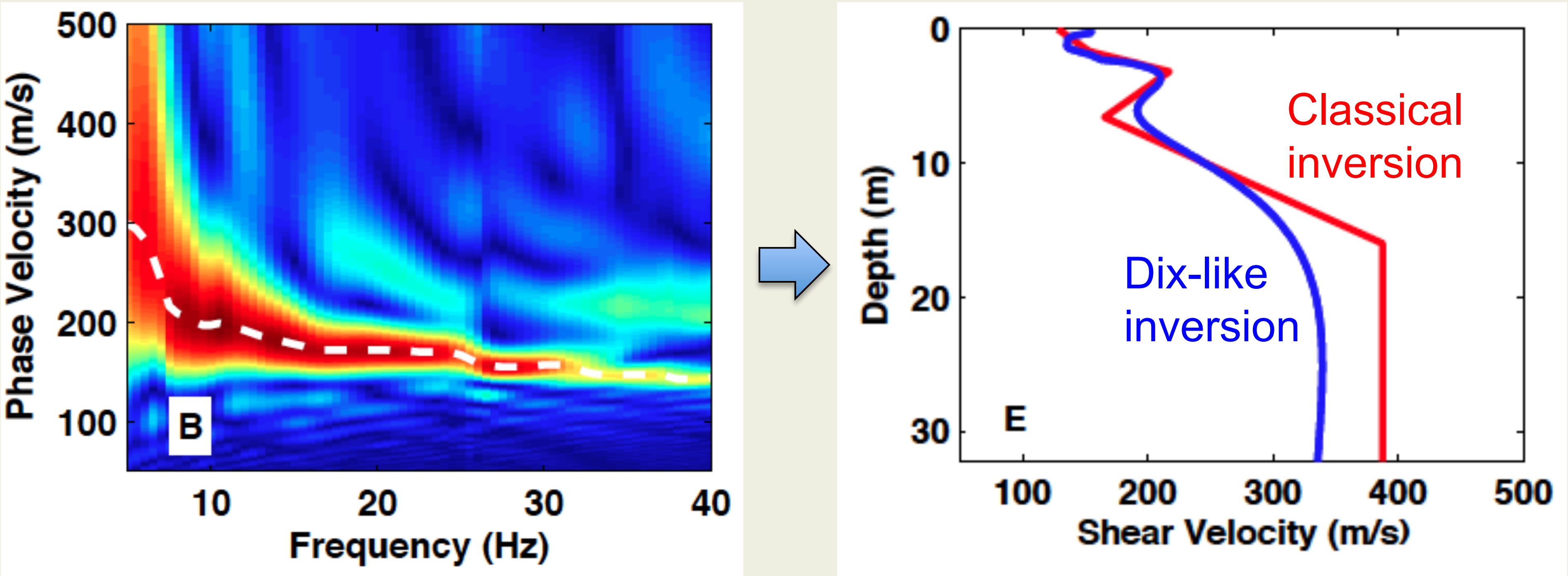
$$I_1 = \frac{1}{2}\int_0^\infty \rho\left(r_1^2 + r_2^2\right)dz \qquad I_2 = \frac{1}{2}\int_0^\infty \left[\left(\lambda + 2\mu\right)r_1^2 + \mu r_2^2\right]dz$$

- $\mu(z) = \rho(z)\beta^2(z)$, etc… then $c_m^2 = \dfrac{\int F_1[z,k_m]\beta^2(z)\,dz}{\int F_2[z,k_m]\,dz}$

- Or $\vec{c}^{\,2} = \mathbf{G}\vec{\beta}^2$, i.e. a linear model for $\beta^2(z)$!

Examples

- Shallow shear-wave velocity inversion



- Reproduces main features, including 2 velocity reversals
- Much faster! 0.1s vs. minutes (on laptop)
- *Errors dominated by data, not inversion technique*!

Examples

- Shallow shear-wave velocity inversion
- 9-line code!

```
fr = [10 15 20 30 40 50 60 70 80];
c = [550 450 300 220 185 175 170 166 164];
z = [0 1 2 3 4 5 6 7 8 inf];
w = 2*pi*fr; k = w./c; kz = k'*z;

f_ray = -131.60*exp(-1.8362*kz)+6.9595*exp(-1.7556*kz)
+274.10*exp(-1.7380*kz) ...
   -12.233*exp(-1.6859*kz)-185.38*exp(-1.6750*kz)
+1.7184*exp(-1.6574*kz) ...
   -143.57*exp(-1.6398*kz)+344.66*exp(-1.6053*kz)
+4.8068*exp(-1.5877*kz) ...
   -160.68*exp(-1.5356*kz);
G = diff(f_ray,1,2);
beta_sq = (G'*G+0.01*eye(length(z)-1))^-1*G'*c.^2';
beta = sqrt(beta_sq);
plot(z(1:length(z)-1),beta,'x-');
```

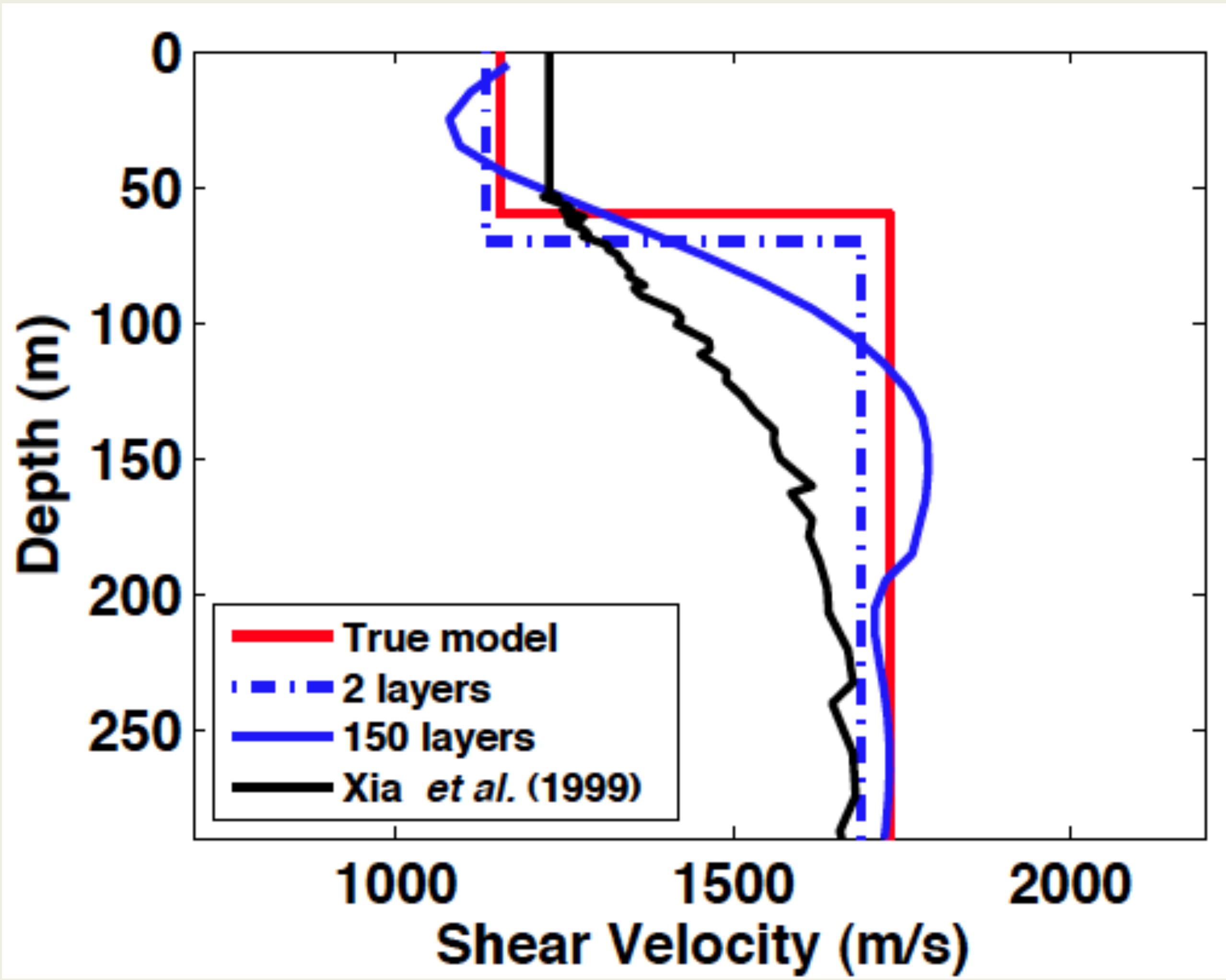
Note: Code with ‘good’ regularization is slightly longer (~20 lines): See paper

Examples

- Shallow shear-wave velocity inversion
- 2-layer model

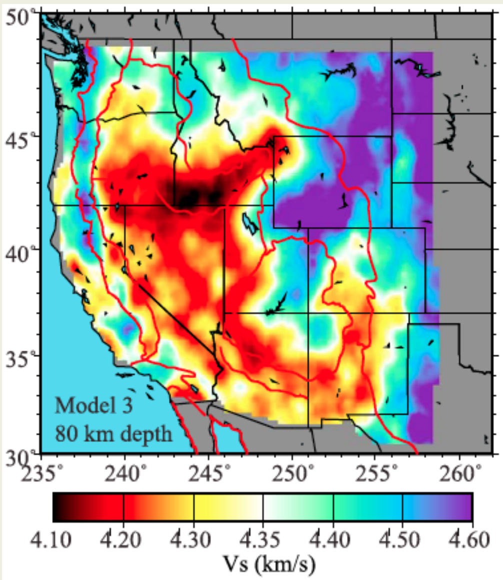
- Improvement over previous approximations

- Over-param. inversion also works

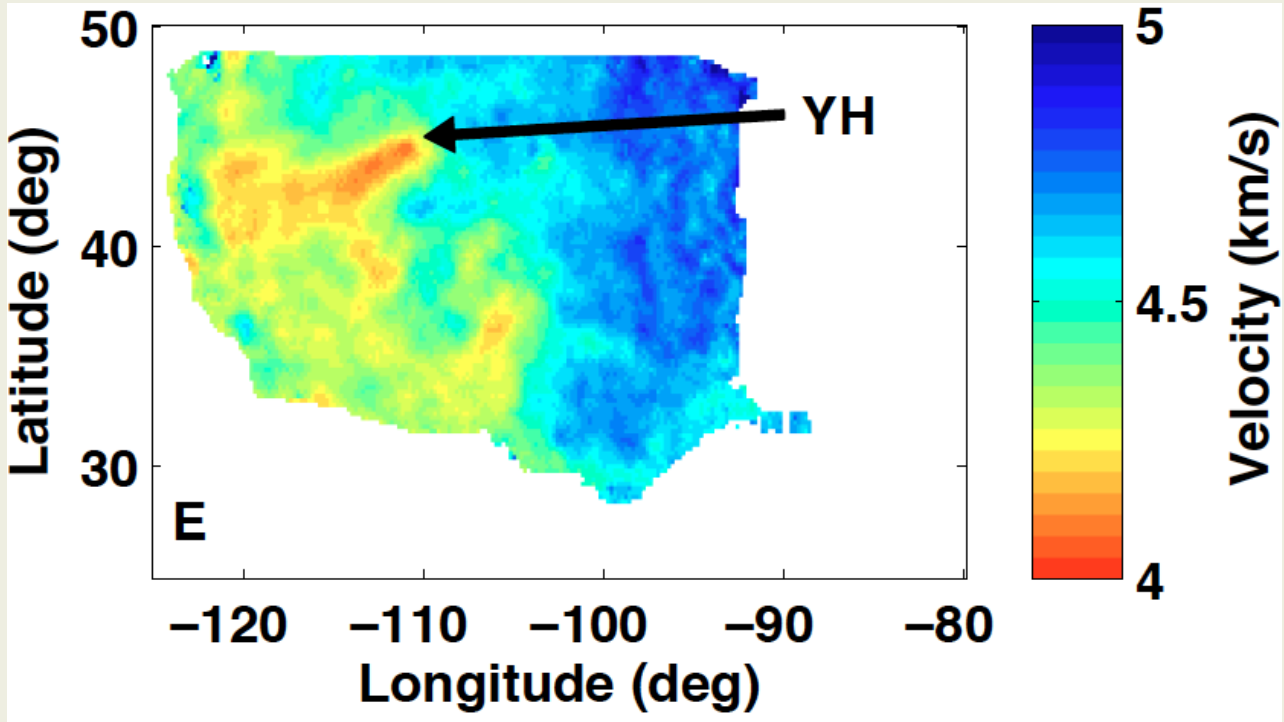


Examples

- USArray-scale crustal inversion (upper mantle velocities)



Lin, Schmandt, Tsai, 2012

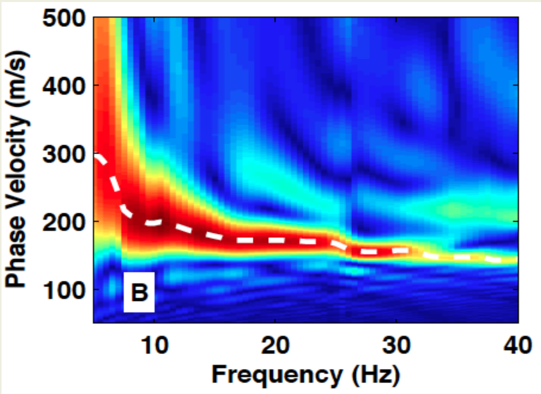


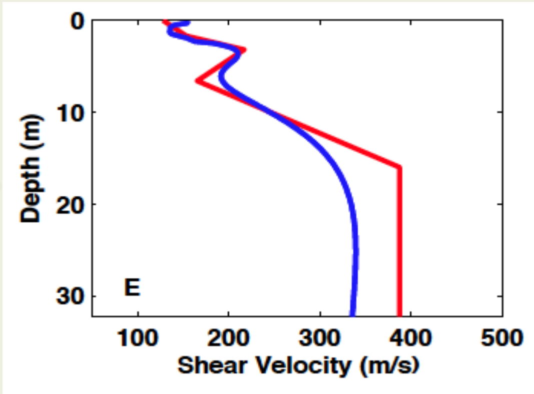
- Left: Classical, using 14 phase + 8 H/V measurements
- Right: Under-parameterized Dix-like inversion
  - uses only 3 phase meas. (8,20,40s); solves for v_{crust}, v_{mantle}, H_{crust}

Inversion Conclusions

- There is a **Dix-equation analog for surface waves**
  - Relates shear-wave velocities $V_s(z_m)$ to phase velocities $c(\omega_n)$
- Inversion is **linear**, **fast**, and **accurate**
- Code is short and **easily understood**
- Works well at local (30m) and crustal (50km) scales

Haney & Tsai, *Geophysics*, 2015, in revision





Conclusions

- Interferometric theories can be intuitively related
- Alternative surface-wave inversion can be useful

- Now, let's go to the beach!