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On dynamic sliding with rate- and state-dependent friction laws

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SUMMARY

We consider the dynamic motion of an elastic slab subject to non-linear friction on a rigid substratum. We consider two categories of friction laws. The first corresponds to *rate-dependent models*. This family of models naturally contains the steady-state models of Dieterich and Ruina. In the second category—*regularized rate-dependent models*—the friction law is mainly rate-dependent, but it is more complex and it is defined by a general differential relation. The regularized rate-dependent models include as a particular case the classical rate and single-state variable friction laws of Dieterich and Ruina. The two models of friction show very different mathematical behaviour. The rate-dependent models lead to a scalar equation, which has no unique solution in general. If the velocity weakening rate exceeds a certain value, many solutions exist. To overcome this difficulty, we have to define a formal rule of choice of the solution. To discriminate between solutions we propose using the perfect delay convention of the catastrophe theory. The second category of models, that is, the regularized rate-dependent models, leads to a differential equation, which has a unique solution. We give its condition of stability and we show that it corresponds to the condition of non-uniqueness of the first model. Considering the particular regularized rate-dependent model of Perrin *et al.* (1995), we show numerically that the limit solution when the characteristic slip $L \rightarrow 0$ is the one corresponding to the rate-dependent model (the steady-state model) assuming the perfect delay convention. Hence, the perfect delay convention takes on a physical sense because it leads to a solution that is the limit of a regular problem. We suggest that the perfect delay convention may be used when pure rate (or mainly rate) dependence is involved. Finally, we analyse briefly the role of the other parameters, A and B, of the rate and state formulation in the context of the shearing slab.

Key words: elastodynamics, friction, non-uniqueness, perfect delay convention, rate dependence, regularization.

1 INTRODUCTION

The rupture process in earthquakes has a spatio-temporal complexity. Among the causes of that complexity, the heterogeneities of behaviour of the material and the heterogeneities of stress inside the faults as well as the geometry of the faults play a major role, but it has been proposed that the non-linear friction on the faults is essential to understand that complexity (see Carlson & Langer 1989). Meanwhile, if one includes non-linearities in a fault model, one rarely knows if the problem is well posed because of the presence of discontinuities or singularities induced by the model. Indeed, theoretical results on the nature of the discontinuities are missing. In this paper we only focus on the consequence of using a non-linear rate-dependent friction law. Here simple models are very useful for that purpose. We can consider that a simple model is based on the definition of two things: the behaviour of the body (i.e. the crust) and the type of friction on the interface (i.e. the fault). As far as the body is concerned, the block slider model has already been well studied in the past few years (see Burridge & Knopoff 1967; Scholz 1990; Gu *et al.* 1984; Rice & Ruina 1983). In this paper we study the sheared slab, which is a model based on the equations of elastodynamics in one dimension. Concerning the friction law, Dieterich and Ruina laws (see Ruina 1983 for earlier works or see Dieterich 1994 for further models) and their derived laws (see Perrin *et al.* 1995) have been proposed and experimented on in the laboratory (see Chester 1995). Campillo *et al.* (1996) showed that the choice of the body and especially the choice between a rigid block and an elastic slab is fundamental since the theoretical behaviour observed is very different mathematically and physically speaking in the case of a non-linear rate-dependent friction law. They proved that an elastic slab combined with rate-dependent friction leads to a problem with multiple solutions and subsequent temporal discontinuities of the sliding velocity (shocks). Since the rate and state friction laws are mainly rate-dependent, our aim is to understand more precisely the relationship between two categories of models, both pure rate-dependent friction models, and rate- and state-dependent friction models. We write down the constitutive equations of the slab and we consider a more general formulation for both models of friction. We examine existence, uniqueness and stability of the solution for both types of models. The first category of models exhibits a problem of uniqueness of the solution if the velocity weakening exceeds a certain value. We propose to use the perfect delay convention to choose the solution. The second model has a unique solution and we characterize its stability in a classical way. We perform numerical simulations with the Perrin *et al.* (1995) model in order to show that the second model is able to converge to the first model solved by the perfect delay convention. Finally, we discuss the opportunity of using the perfect delay convention to solve problems involving rate-dependent friction laws in elastodynamics.

2 PROBLEM STATEMENT

Let us consider the 1-D shearing of an infinite linear elastic slab bounded by the planes $x=0$ and $x=h$. The slab is in frictional contact with a rigid body at $x=0$ (see Fig. 1). Assume that the displacement field is zero with respect to the direction Oz and that it depends only on $x\in[0, h]$. At the upper surface of the layer we impose a constant and uniform compressive normal stress σ (i.e. $\sigma_{xx}(t, h)=-\sigma$) and a tangential constant velocity V (i.e. $\dot{u}_y(t, h)=V$). Since we are interested only in the evolution of the displacement u_y and of the stress σ_{xy} we shall assume that the displacement u_x is a linear function of the coordinate x and that it can be obtained from Hooke's law.

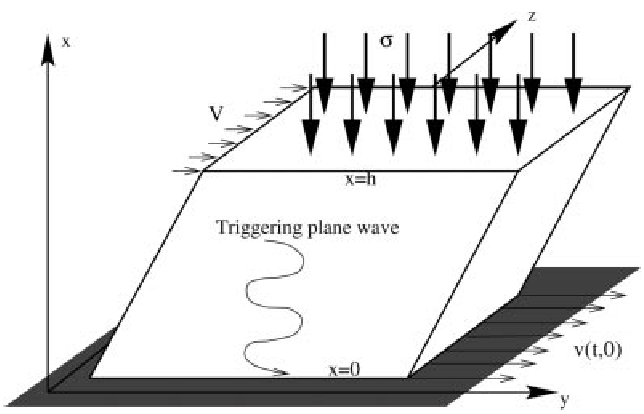

Figure 1. The model of the slab dragged on $x=h$ with velocity V and slipping on $x=0$ with velocity $v(t, 0)$. The frictional condition at $x=0$ is proportional to the normal stress σ. The instability phenomenon is triggered by the initial conditions such as a plane wave.

For simplicity we denote $u=u_y$, $v=\partial u_y/\partial t$ and $\tau=\sigma_{xy}$; the momentum balance law and Hooke's law may then be written as follows:

$$\left.\begin{aligned} \rho \frac{\partial v}{\partial t}(t, x) &= \frac{\partial \tau}{\partial x}(t, x) \quad (1)\\ \frac{\partial \tau}{\partial t}(t, x) &= \mu \frac{\partial v}{\partial x}(t, x) \quad (2)\end{aligned}\right\} \quad \forall (t, x)\in]0, T]\times]0, h[\,,$$

where ρ is the density and μ the elastic shear coefficient. For simplicity we limit our study on $t\in]0, T]$, where T is the time of reflection on the upper boundary of the wave created by the slip itself, i.e. $T=2h/\beta$, where $\beta=\sqrt{\mu/\rho}$ is the shear wave speed.

The boundary condition for $x=h$ and $t\in[0, T]$ is as follows:

$$v(t, h)=V\,. \quad (3)$$

Since we limit our study to a duration less than or equal to T (the time of reflection on the upper boundary), the thickness h of the slab plays no role here; only the boundary condition at $y=0$ plays a role.

On the friction surface $x=0$ we consider two types of non-linear relationships between the shear stress $\tau=\tau(t, 0)$ and the shear motion $v=v(t, 0)$ at the contact. These relations include other variables needed to describe the phenomenon of friction observed in many experiments. In both models of friction, τ is proportional to the load σ at the upper surface of the body and we assume that the slip rate v and the stress τ are always positive. The two models we consider are a pure rate-dependent friction and a regularized rate-dependent friction that includes the classical Dieterich and Ruina model with a single-state variable as a particular case.

In the *rate-dependent models*, we study the general case of a rate-dependent friction defined by

$$\begin{aligned} &\tau=\tau_v(v) \quad \text{for} \quad v>0\,,\\ &\tau\leq\tau_s \quad \text{for} \quad v=0\,, \end{aligned} \quad (4)$$

where the function $\tau_v(v)$ is any continuous and decreasing function of the slip rate and $\tau_s=\tau_v(0)$.

In the *regularized rate-dependent models*, we study a family of friction laws defined as follows:

$$\begin{aligned} &\dot{\tau}=F(v)\dot{v}-K(v)H(\tau-\tau_v(v)) \quad \text{for} \quad v>0\,,\\ &\tau\leq\tau_s \quad \text{for} \quad v=0\,, \end{aligned} \quad (5)$$

where K, H and F are explicit continuous functions of the slip rate such that $K(v)>0$, $H(0)=0$, $H'(v)>0$ and $F(v)>0$. The above formulation has the advantage of leaving the choices for the functions F, K, H and τ_v completely free.

First, we introduce the Perrin *et al.* (1995) model, which is given by

$$\begin{aligned} &\tau(t)=\tau_0+A\ln((V_0+v)/(V_\infty+v))+B\ln(1+\theta(t)(V_\infty-V_0)/L)\,,\\ &\dot{\theta}(t)=1-\theta(t)(V_0+v)/L\,. \end{aligned} \quad (6)$$

This formulation is related to the Dieterich–Ruina slowness model (see Ruina 1983). In this formulation, the state variable θ is allowed to evolve during the stick phase and its steady-state value θ_{ss} is bounded in $[0, L/V_0]$. Besides the state-variable evolution, this friction law involves two instantaneous velocity dependences: a velocity weakening and a velocity strengthening. The slowness model is in sharp contrast to the initial Ruina–Dieterich slip model (see Ruina 1983), for which the state variable is frozen during the stick phase.

We now show that the Perrin *et al.* (1995) model is contained in our formulation (5). Indeed, if we take the following functions:

$$F(v)=A\,\frac{V_\infty-V_0}{(V_0+v)(V_\infty+v)}\quad\text{with}\quad A\geq 0\quad\text{and}\quad V_\infty\geq V_0\,,$$

$$K(v)=(V_0+v)/L\,,$$

$$H(\gamma)=B(1-e^{-\gamma/B})\quad\text{with}\quad B\geq 0\,,$$

$$\tau_v(v)=\tau_{ss}(v)=\tau_s+(A-B)\ln\frac{1+v/V_0}{1+v/V_\infty}\quad\text{with}\quad A-B\leq 0\,,\qquad(7)$$

where $\tau_s=\tau_0+(A-B)\ln(V_0/V_\infty)$, we obtain exactly the form proposed by Perrin *et al.* (1995) (6).

As seen previously, the Perrin *et al.* (1995) law, which will be considered in this paper as the reference law, can be written in the form (5). We point out that the choice of the slowness form of Perrin *et al.* (1995) (6) is not really of importance for our work. Indeed, as far as we know, all the classical laws involving a single-state variable can be written in the form (5). We have tested several laws proposed by Dieterich and Ruina (see Ruina 1983) and Perrin *et al.* (1995). This new general formulation (5) shows that the evolution of the stress is governed basically by an instantaneous viscous friction $\tau_v(v)$ (weaker at high velocity) and a regular stress decay governed by $\tau-\tau_v(v)$. During a given step of velocity from V_1 to V_2, the stress decay takes the form of an exponential whose characteristic slip is $V_2/(K(V_2)H'(0))$. In fact, the term $K(v)H(\tau-\tau_v(v))$ can be seen as a viscoplastic regularization of the instantaneous viscous friction $\tau_v(v)$.

Finally, we complete the problem statement by taking the initial conditions,

$$v(0,x)=v_0(x)\,,\quad \tau(0,x)=\tau_0(x)\,,\qquad(8)$$

where v_0 and τ_0 are assumed to be continuous functions.

3 THEORETICAL ASPECTS

Our goal in this section is to analyse the nature of the solution for the two models of friction. First, we recall that the evolution of the velocity and the stress functions on the interface $x=0$ is directly related to the properties of the elastodynamic equations (1) and (2). Indeed, eqs (1) and (2) lead to the 1-D wave equation

$$\frac{\partial^2 v}{\partial t^2}(t,x)=\beta^2\,\frac{\partial^2 v}{\partial x^2}(t,x)\,.$$

The solution of this equation can be easily deduced in the form $v(p,s)=v_+(p)+v_-(s)$ with $p=x+ct$ (downgoing wave) and $s=x-ct$ (upgoing wave). Using these new coordinates p and s, one can easily deduce another property of eqs (1) and (2), namely

$$\frac{\partial}{\partial s}\,[v(p,s)\sqrt{\rho\mu}+\tau(p,s)]=0\,.\qquad(9)$$

Therefore, the expression differentiated in eq. (9) is only a function of the variable p. Hence, it is constant for a given characteristic line $\{(x,t)\in\mathbb{R}^2;\,x+\beta t=p=\text{const}\}$. If we consider the characteristic line $p=0$ and if we denote $\Lambda(t)=v(0,\beta t)\sqrt{\rho\mu}+\tau(0,\beta t)=v_0(\beta t)\sqrt{\rho\mu}+\tau_0(\beta t)$, then we deduce that for $x=0$ and $t\in\,]0,T]$,

$$v(t,0)\sqrt{\rho\mu}+\tau(t,0)=\Lambda(t)\,.\qquad(10)$$

Here $\Lambda(t)$ may be interpreted as a loading (for Λ increasing) or an unloading (for Λ decreasing) on the fault.

This last equation (10) is similar to the one governing the slip along a plane embedded in an infinite elastic medium under an anti-plane shear (see Madariaga & Cochard 1994) in the case of uniform slip. The governing equation gives the resulting shear stress $\tau(t,0)$ as the sum of a local instantaneous radiation term $-v(t,0)\sqrt{\rho\mu}$ and a loading term related to the history of the non-local elastic interactions in both space and time (this does not exist in our study because the slip is uniform) and the external load $\Lambda(t)$. This external load $\Lambda(t)$ is deduced from the initial conditions in the body and it can simulate either a uniform load or an incoming wave. Finally, we note that the previous scalar equation (10) on the friction surface was deduced from the elastodynamic equations, hence it is independent of the choice of friction law.

3.1 Rate-dependent model

By considering eqs (10) and (4) we find the following scalar equations for the slip rate $v(t,0)$ on the frictional interface $x=0$ and $t\in\,]0,T]$:

$$\tau(t,0)=\Lambda(t)\quad\text{for}\quad v(t,0)=0\quad\text{and}\quad\Lambda(t)\leq\tau_s\,,$$

$$\lambda(v(t,0))=\Lambda(t)\quad\text{for}\quad v(t,0)>0\,,\qquad(11)$$

where the function λ is given by

$$\lambda(v)=v\sqrt{\rho\mu}+\tau_v(v)\,.\qquad(12)$$

From eq. (11), it appears that the solution $v(t,0)$ is obtained at the intersection between the curve $\lambda(v(t,0))$ and the horizontal straight line $\Lambda(t)$ (see Fig. 2). Two qualitative types of behaviour are possible according to whether or not λ is monotonic. In the first case—regular behaviour—λ is increasing, i.e. $\tau_v'(v)>-\sqrt{\rho\mu}$, and there is a unique continuous solution $v(t,0)$ for all continuous $\Lambda(t)$. In the second case—irregular

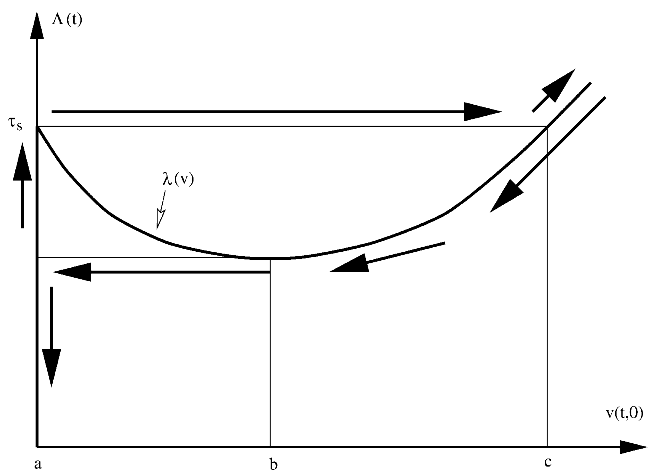

Figure 2. Solution of the rate-dependent model assuming the perfect delay convention. The solution of the scalar equation $\lambda(v(t,0))=\Lambda(t)$ follows a path of hysteresis (with arrows) in the phase plane $(v(t,0),\Lambda(t))$.

behaviour—λ is no longer monotonically increasing (see Fig. 2). For instance, let us suppose that there exists $[a, b] \subset R^+$ such that λ is decreasing on $[a, b]$, i.e. $\tau_v'(v) < -\sqrt{\rho\mu}$ for all $v \in [a, b]$. The solution $v(t, 0)$ is unique if $\Lambda(t) < \lambda(b)$ or $\lambda(a) < \Lambda(t)$, but *there are two or three solutions* for $\lambda(b) \leq \Lambda(t) \leq \lambda(a)$.

However, since the solution of the problem is not uniquely determined during the irregular behaviour, we need a criterion to select the more appropriate physical solution among these three possibilities. *Whatever selection rule is chosen to discriminate between solutions, shocks will occur.* A possible choice for this criterion is the *'perfect delay' convention*: the system only jumps when it has no other choice (see Ionescu & Paumier 1994 and Campillo *et al.* 1996). In this way different paths of solutions are obtained in acceleration and deceleration processes and hysteresis occurs. Such a process is illustrated in Fig. 2. Hence, the physical solution $v(t, 0)$ represented in Fig. 3 maximizes its interval of continuity. Madariaga & Cochard (1994) used these paths in the context of 2-D antiplane elastodynamics but they did not refer explicitly to the perfect delay convention as an additional criterion of choice. One consequence is that $v(t, 0)$ never follows the decreasing branch, for which $\tau_v'(v(t, 0)) < -\sqrt{\rho\mu}$, i.e. $v(t, 0) \notin [a, b]$ (see Figs 2 and 3).

If we consider $\tau_v(v)$ defined by analogy with the Perrin *et al.* (1995) model by $\tau_v(v) = \tau_{ss}(v)$ (see eq. 7), then the irregular behaviour is present if

$$\sqrt{\rho\mu} < (B - A)\,\frac{V_\infty - V_0}{V_0 V_\infty}\,. \tag{13}$$

It follows from Ionescu & Paumier (1994) that the perfect delay convention *is not related to a simple energy criterion*. One notices that the delay criterion, which comes from the catastrophe theory (see e.g. Poston & Stewart 1978), is implicitly present in the analysis of many physical problems. Generally speaking it is used in the study of static or quasi-static problems (see Ionescu & Paumier 1996) and it is justified by a dynamic stability analysis, that is, the (static or quasi-static) position chosen by the perfect delay convention is always a stable position. The use of this criterion in our context, even if it is intuitively reasonable, has no justification because our study is already fully dynamic. We show in the next section that this choice can be motivated by the analysis of a regularized rate-dependent model.

3.2 Regularized rate-dependent model

By taking into account eqs (10) and (5) we find the equations for the slip rate on the frictional interface $x = 0$ for all $t \in]0, T]$:

$$\tau(t, 0) = \Lambda(t) \quad \text{for} \quad v(t, 0) = 0 \quad \text{and} \quad \Lambda(t) \leq \tau_s\,,$$

$$\dot{v}(t, 0) = \frac{\dot{\Lambda}(t) + K(v(t, 0))H[\Lambda(t) - \lambda(v(t, 0))]}{\sqrt{\rho\mu} + F(v(t, 0))} \quad \text{for} \quad v(t, 0) > 0\,. \tag{14}$$

If we add to the first-order differential equation (14) the initial condition $v(0, 0) = v_0(0)$, we obtain a Cauchy problem that has a unique smooth solution. This implies that the regularized rate-dependent model corresponds to a well-posed mathematical problem. This is not true of the rate-dependent model in the irregular behaviour, that is, when the condition of non-uniqueness $\tau_v'(v(t, 0)) < -\sqrt{\rho\mu}$ is verified.

In order to give a linear stability analysis of our problem, we consider a small initial perturbation $\tilde{v}_0$, $\tilde{\tau}_0$ of the steady sliding

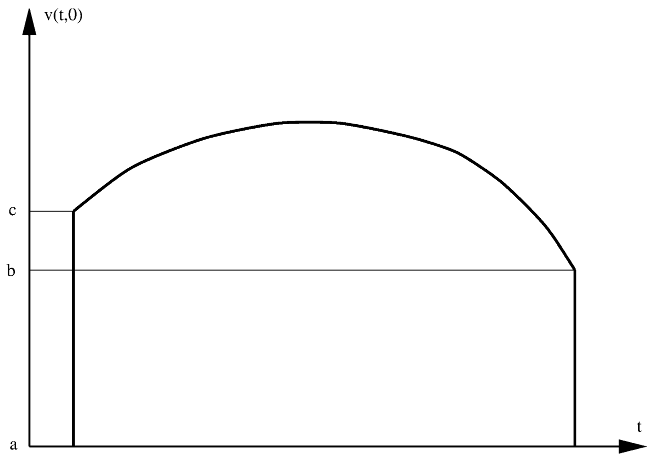

Figure 3. The sliding velocity of the rate-dependent model, assuming the perfect delay convention, makes two jumps in the irregular case. The first is from $v(t, 0) = a$ to $v(t, 0) = c$ and the second is from $v(t, 0) = b$ to $v(t, 0) = a$.

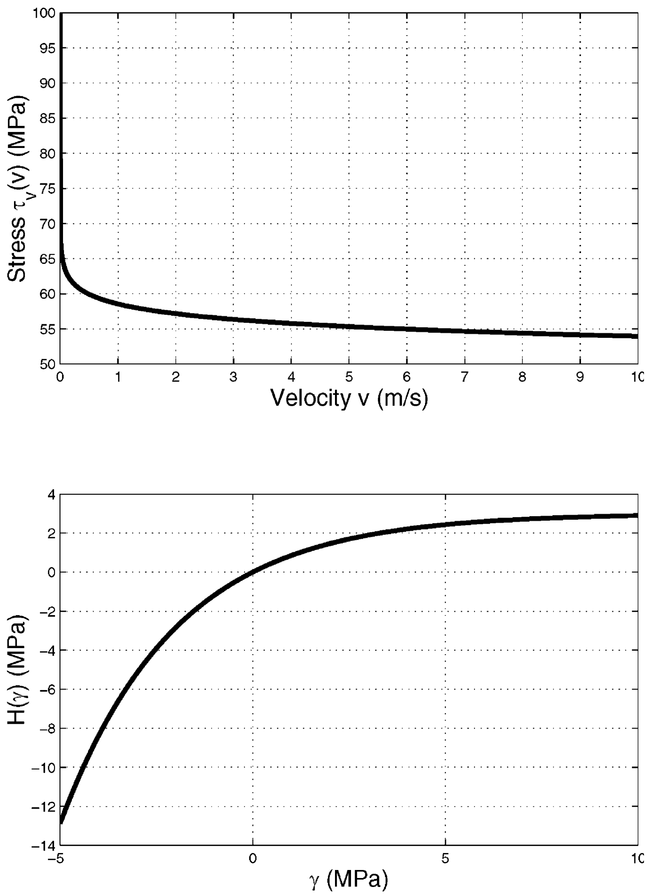

Figure 4. The rate and state friction law of Perrin *et al.* (1995) is mainly characterized by a rate-dependent friction $\tau_v(v)$ and its function H that operates the regularization. We verify that $H(0) = 0$ and $H'(0) = 1 > 0$. The numerical values are $V_0 = 1.0 \times 10^{-9}$ m s^{-1}, $V_\infty = 1.0 \times 10^{9}$ m s^{-1}, $\tau_s = 100$ MPa and $B - A = 3$ MPa.

rigid motion. The slab slides uniformly at a constant velocity V. In this case the initial conditions are $v_0(x)=V+\tilde{v}_0(x)$, $\tau_0(x)=\tau_v(V)+\tilde{\tau}_0(x)$. By denoting $\tilde{\Lambda}(t)=\tilde{\tau}_0(\beta t)+\sqrt{\rho\mu}\tilde{v}_0(\beta t)$ we obtain $\Lambda(t)=\lambda(V)+\tilde{\Lambda}(t)$. If we put $v(t, 0)=V+\tilde{v}(t)$ then from (14) we deduce

$$\dot{\tilde{v}}(t)=f(\tilde{v}(t), \tilde{\Lambda}(t), \dot{\tilde{\Lambda}}(t)), \tag{15}$$

where

$$f(\tilde{v}, \tilde{\Lambda}, \dot{\tilde{\Lambda}})=\frac{\dot{\tilde{\Lambda}}+K(V+\tilde{v})H[\lambda(V)+\tilde{\Lambda}-\lambda(V+\tilde{v})]}{\sqrt{\rho\mu}+F(V+\tilde{v})}. \tag{16}$$

The first-order instability condition reads $\partial_v f(0, 0, 0)>0$, which is equivalent to $\lambda'(V)<0$ or to $\tau_v'(V)<-\sqrt{\rho\mu}$.

As in the rate-dependent model, if λ is decreasing on $[a, b]$ (see Fig. 2), the motion is an unstable rigid steady sliding for $V\in[a, b]$. We remark that if the solution of the rate-dependent model is chosen by using the perfect delay convention, then the interval of non-existence $[a, b]$ for the rate-dependent model corresponds to a domain of unstable evolution for the regularized rate-dependent model.

4 DISCUSSION

To illustrate our discussion, we perform a set of numerical simulations with the Perrin *et al.* (1995) formulae with parameters complying with the condition of instability (13). We recall that this choice of parameters corresponds to the condition of non-uniqueness in the first model (13). We perform a cycle of loading in order to observe the whole path followed by the system, i.e. a stick–slip–stick sequence. We take some initial conditions, consisting of the superposition of a static load τ_0^0 ($V=0$ at the upper edge) and an arriving plane wave defined by its stress amplitude τ^0 and its period $T_{\rm load}$. The initial condition corresponding to this load can be written $\tau_0(x)=\tau_0^0+\tau^0/2\sin(\pi x/(\beta T_{\rm load}))$ and $v_0(x)=\tau^0/(2\sqrt{\rho\mu})\sin(\pi x/(\beta T_{\rm load}))$, so that the load is $\Lambda(t)=\tau_0^0+\tau^0\sin(\pi t/T_{\rm load})$.

We recall that h is large enough that it plays no role in this study. In all the simulations we take the following constant values: $\mu=34.3$ GPa, $\rho=2800$ kg m^{-3}, $V_0=1.0\times10^{-9}$ m s^{-1}, $V_\infty=1.0\times10^{9}$ m s^{-1} (no cut-off of the friction for great sliding velocity), $\tau_s=100$ MPa, $\tau_0^0=0.9\tau_s$, $\tau^0=0.36\tau_s$, $T_{\rm load}=8$ s.

In the previous sections we showed that the two models of friction lead to very different mathematical behaviour. Indeed, the rate-dependent model induces problems of non-uniqueness and discontinuities, unlike the regularized rate-dependent model, which has a unique and continuous solution. However, we have also seen that the domain of discontinuity of the velocity in the solution of the rate-dependent model in the irregular case corresponds to the domain of (continuous) instability of the regularized rate-dependent model. We now want to show how our intuitive choice of the perfect delay convention is justified. In order to do this we study the limit solution of the regularized rate-dependent model when $K(v)\to+\infty$ and we compare these results with the results of the rate-dependent model assuming the perfect delay convention. To this end we perform a series of numerical studies with the Perrin *et al.* (1995) formulae for $L=1$, 0.1 and 0.01 m with $A=0$ and $B=2.3$ MPa. The corresponding functions $\tau_v(v)$ and H of the Perrin *et al.* (1995) model are plotted in Fig. 4. The results are presented in Figs 5 and 6. We observe that *the*

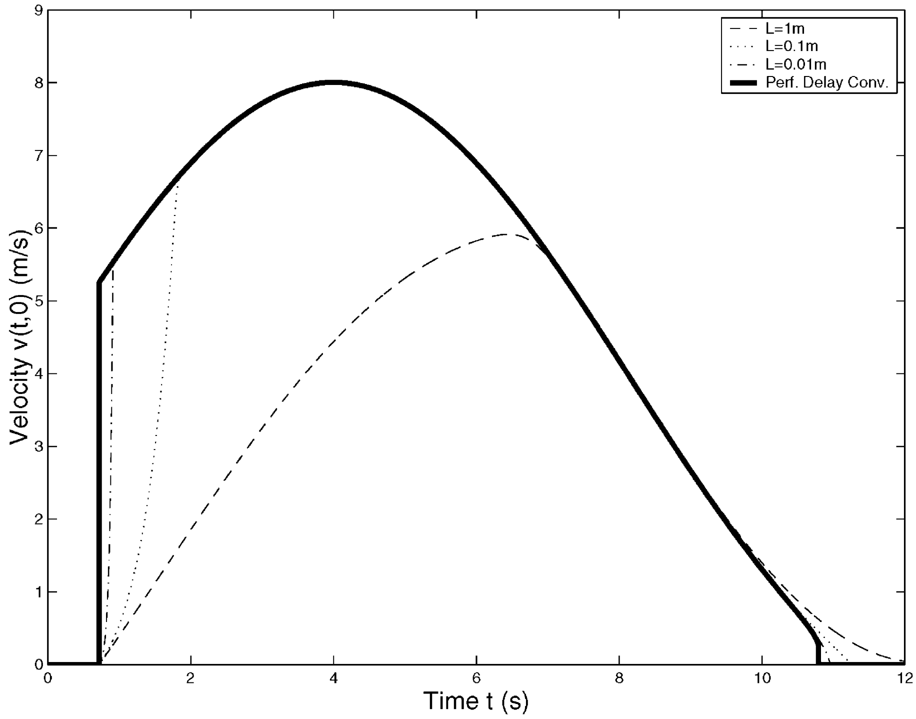

Figure 5. Plots of $v(t, 0)$ for different values of L in the regularized rate-dependent model using the Perrin *et al.* (1995) formulae. One observes the numerical convergence of the solution of the regularized rate-dependent model to the solution of the rate-dependent model assuming the perfect delay convention. For $L=1$ cm, the limit has already been reached and the regular jumps look like discontinuities. The numerical values are $\mu=34.3$ GPa, $\rho=2800$ kg m^{-3}, $V_0=1.0\times10^{-9}$ m s^{-1}, $V_\infty=1.0\times10^{9}$ m s^{-1}, $\tau_s=100$ MPa, $A=0$, $B=2.3$ MPa, $\tau_0^0=0.9\tau_s$, $\tau^0=0.36\tau_s$ and $T_{\rm load}=8$ s.

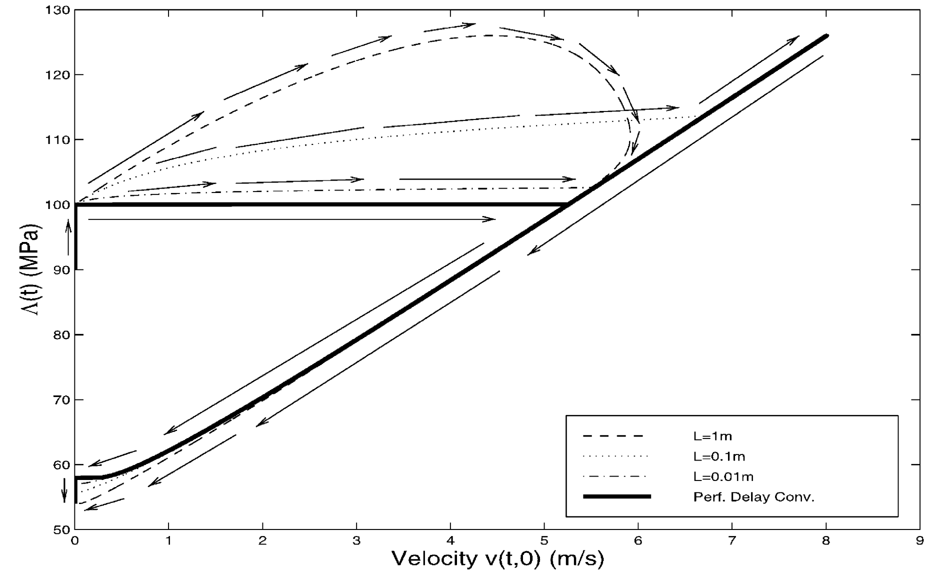

Figure 6. The plots of the trajectory in the phase plane ($v(t, 0)$, $\Lambda(t)$) for the simulations of Fig. 5. One observes how the trajectory converges for small L to the trajectory of the rate-dependent model assuming the perfect delay convention. The numerical values are $\mu=34.3$ GPa, $\rho=2800$ kg m^{-3}, $V_0=1.0\times10^{-9}$ m s^{-1}, $V_\infty=1.0\times10^{9}$ m s^{-1}, $\tau_s=100$ MPa, $A=0$, $B=2.3$ MPa, $\tau_0^0=0.9\tau_s$, $\tau^0=0.36\tau_s$ and $T_{\rm load}=8$ s.

rate and state model converges exactly to the rate-dependent model (i.e. the steady-state model) solved with the perfect delay convention when $L\to0$. We see in Fig. 5 precisely how the smooth jumps in the Perrin *et al.* (1995) model converge to the sharp jumps in the steady-state model. Fig. 6 illustrates the convergence of the path of the velocity $v(t, 0)$ in the phase plane ($v(t, 0)$, $\Lambda(t)$).

As we stated in the last section, the problem of non-uniqueness in the pure rate-dependent model cannot be solved in a classical way; another physical condition must be added to the formulation. We propose that taking the limit of a convenient well-posed problem gives us the necessary constraint. Finally, we observe that *taking the limit solution is identical to using the perfect delay convention as an*

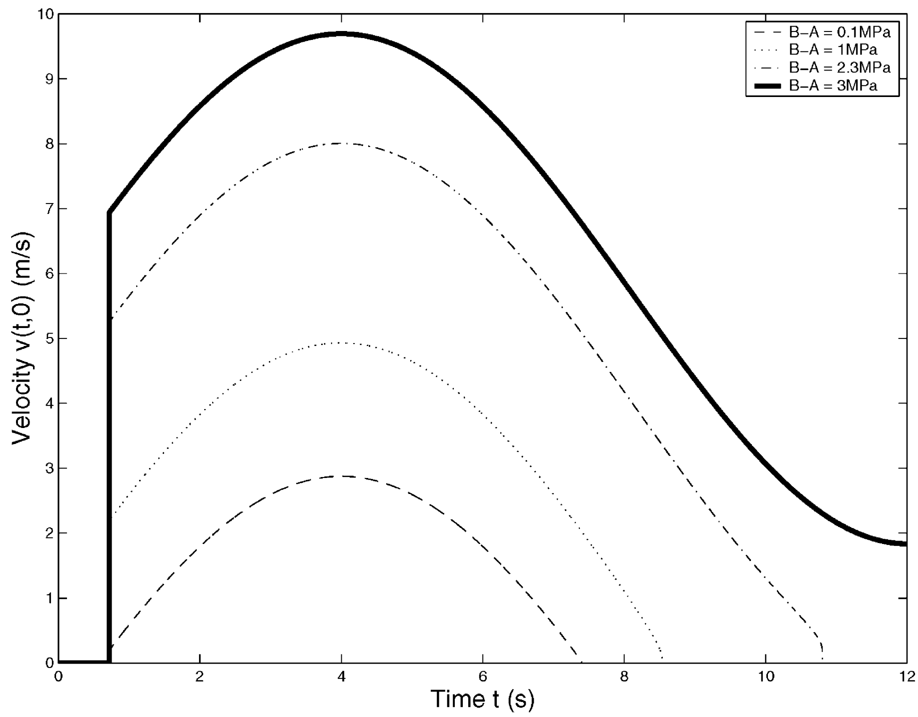

Figure 7. The effect of $B-A$ in the Perrin *et al.* (1995) model in the context of the slab. One observes that $B-A$ gives only the size and the existence of the two velocity jumps. For large values of $B-A$ the second jump may disappear and consequently the system does not stick again. The numerical values are $\mu=34.3$ GPa, $\rho=2800$ kg m^{-3}, $V_0=1.0\times10^{-9}$ m s^{-1}, $V_\infty=1.0\times10^{9}$ m s^{-1}, $\tau_s=100$ MPa, $A=0$, $L=0.1$ mm, $\tau_0^0=0.9\tau_s$, $\tau^0=0.36\tau_s$ and $T_{\rm load}=8$ s.

additional condition. It gives significance to this criterion in dynamical problems. This very simple result has, however, great importance, since many fault models are mainly rate-dependent and cannot be solved in an ordinary way. Indeed, if one takes the initial problem statement, the scalar equation (11) is hidden. In the irregular case, this equation has many solutions and therefore it cannot be solved numerically without further constraints. Two modes of behaviour are possible when faced with this problem. First, one may observe that pure rate-dependent models are not compatible with elastodynamics and they must be abandoned. Second, if experimental evidence requires the use of these models, one can introduce a rule of choice and a solution to the problem in which shocks exist. Those shocks can be the cause of problems in all resolutions in one or more dimensions. In our analysis we propose a way of solving the problem exactly in the 1-D case. It requires the use of the characteristic line integration and the perfect delay convention. We show that *the rate and state model can be considered as a regularization of the steady-state model*. The parameter L controls the regularization. In the experiments at constant velocity, Scholz (1990) interpreted L as the characteristic slip of the friction, that is, the slip that the stress needs to reach a new steady-state value after a velocity step. Our analysis shows that L can also be interpreted through its effect on the duration of the jumps. Indeed, we can form L/V_0 to obtain a characteristic time (see the dimensional analysis in Perrin *et al.* 1995). In order to apply the above conclusion to cycles of loading as we do, we have to verify that the period of the cycle is much larger than the duration of the jumps (controlled by L). Indeed, in the case of a large number of cycles and a small initial noise, one could not easily conclude that the dynamical non-linear evolution of a pure rate-dependent model is still the limit of a regularized dependent model.

We now illustrate briefly the role of the parameters A and B of the rate and state model in the context of the slab. After our analysis it is clear that we shall study the roles of $B-A$ and A separately. Indeed, $B-A$ appears in the function $\tau_{ss}(v)$ while A alone appears in the function $F(v)$. Thus, we first make simulations in the instability condition with $L=0.1$ mm, $A=0$ and $B-A=0.1$, 1, 2.3 and 3 MPa. The results are presented in Fig. 7. We observe that $B-A$ only gives the size of the (continuous) jumps of the sliding velocity. We note that the second jump does not exist if $B-A$ exceeds a certain limit (see the curve $B-A=3$ MPa) because the load $\Lambda(t)$ does not decrease enough to reach the minimum of the function $\lambda(v)$. We note that this last result depends on the amplitude of the load $\Lambda(t)$.

To illustrate the role of A we perform another simulation with $B-A=2.3$ MPa and $A=0$, 0.1, 1, 2 and 3 MPa. The results are presented in Fig. 8. The role of A is to delay the first jump. Note that A has no role in the second jump; this is reasonable because $F(v)$ is greater at low velocity in the rate and state formulae. Scholz (1990) interpreted A as having an instantaneous effect of strengthening due to the velocity. We observe also that if A exceeds a certain limit (see the curve $A=3$ MPa), the instability does not develop any more since the load has decreased before the instability (i.e. the first jump) has occurred. This effect depends on the type of load $\Lambda(t)$. Indeed, the harmonic wave considered here loads the system but unloads it before the instability develops.

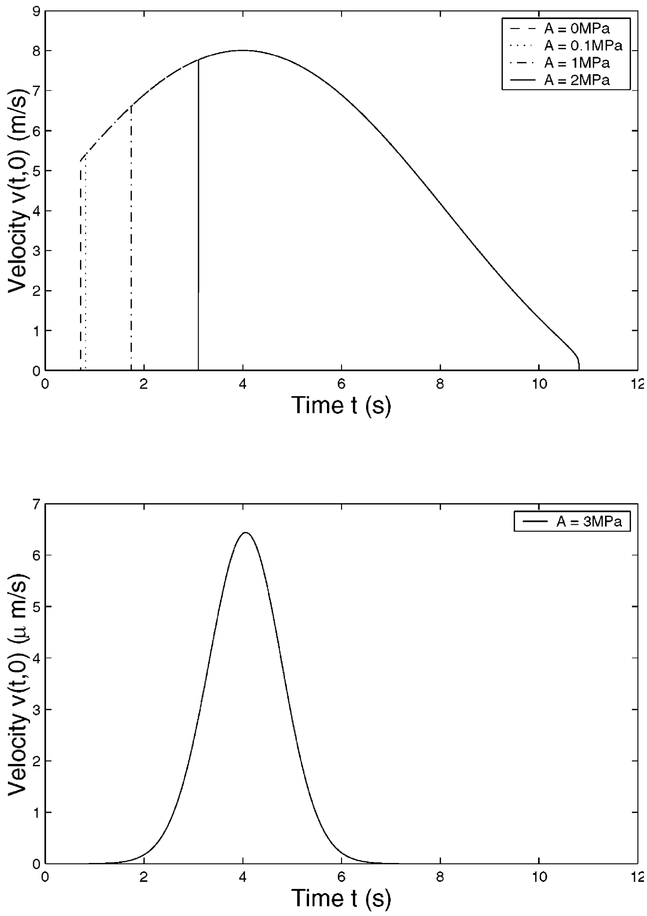

Figure 8. The effect of A in the Perrin *et al.* (1995) model in the context of the slab. One observes that A has a delaying effect on the instability. On the lower figure, we see that large values of A may inhibit the instability phenomenon in the case of load cycles. The numerical values are $\mu=34.3$ GPa, $\rho=2800$ kg m^{-3}, $V_0=1.0\times10^{-9}$ m s^{-1}, $V_\infty=1.0\times10^{9}$ m s^{-1}, $\tau_s=100$ MPa, $B-A=2.3$ MPa, $L=0.1$ mm, $\tau_0^0=0.9\tau_s$, $\tau^0=0.36\tau_s$ and $T_{\mathrm{load}}=8$ s.

5 CONCLUSIONS

We have studied two categories of friction law in the 1-D case of a shearing slab. Our solution is based on the property of conservation along the characteristic lines. The problem of purely rate-weakening friction exhibits non-uniqueness when the weakening rate is larger than a given value. In this case we need a criterion to choose the solution and we propose the perfect delay convention. The second category of friction law we consider is regularized rate-dependent models, in which one finds as a particular case the classical rate- and single-state-variable-dependent models. We have found that the classical notion of stability in the second model is related to the notion of non-uniqueness in the corresponding first model. We have shown that when the characteristic slip $L\rightarrow0$, the solution obtained with the regularized problem converges to that of the purely rate-dependent model assuming the perfect delay convention. The perfect delay convention takes on a physical sense because it leads to a solution that is the limit of a regular problem. These results are of interest in the understanding of fault models with rate-dependent friction because they allow one to solve explicitly the time discontinuities in one dimension, which is impossible with a classical computation. When the same problems of non-uniqueness appear in more general 2- or 3-D fault models, the perfect delay convention could be an important element to use in numerical simulations. We performed a series of numerical experiments to test the sensitivity on the different parameters. The parameter L determines the duration of the velocity jumps, and it appears that for $L=10$ μm the duration is very small compared to the timescale of the total slip event (8 s). The parameter $B-A$ gives the size of the velocity jump directly and the parameter A governs the delay before the instability.

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