Slip-weakening Friction on a Periodic System of Faults: Spectral Analysis

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Abstract

We consider the problem of antiplane shearing on a periodic system of collinear faults under a slip-dependent friction law in linear elastodynamics. A spectral analysis is performed in order to characterize the existence of unstable solutions. The structure of the spectrum for the associated eigenvalue problem is obtained. We prove the monotone dependence of the eigenvalues on the friction parameter. The study the static eigen-problem gives the existence of a limit of stability. The eigenvalue problem is reduced to a hyper-singular integral equation, which is solved through a semi-analytical technique. The general eigen-solution consists of a set of eigenfunctions with a physical periodicity which is a multiple of the natural (geometrical) period of the system. The numerical solution allows us to investigate the behavior of the eigenvalues/functions and to conjecture some specific properties of the spectrum.

1 Introduction

The existence of the earthquake initiation phase, preceding the dynamic rupture, has been pointed out in recent years by detailed seismological observations (e.g. [11], [8]) and has been recognized in laboratory experiments on friction, as reported in [7], [16], [15].

A physical model of slip weakening friction (i.e. the decrease of the friction force with the slip) was introduced by Rabinowicz [17] in the geophysical context of earthquakes modelling and a lot experimental studies (see for instance [16]) pointed out the good agreement of this model with the experimental data. Since the slip-dependent friction model is rate independent and can describe a large variation of the slip rate, it was intensively used in the description of earthquake initiation (see [2, 13, 4, 12, 1]). For a bounded system of faults a stability analysis was performed in [5, 6]. In [5] a threshold of stability was computed for a single fault geometry. This same universal constant was recently obtained in [18] in the case of a nonuniform fault loading.

Our goal here is to study a mathematical problem corresponding to this phase of dynamic and unstable initiation under a slip-dependent friction law on a periodic fault geometry. We shall concentrate on the elastodynamic analysis of the friction in the anti-plane case. More precisely, we focus on the initiation of the shear process during the weakening stage to point out simple mathematical properties of its unstable evolution. The spectral analysis plays a key role in the description the nucleation phase. Indeed, the shape of the eigenfunctions determine the signature of the initiation phase, as showed in [19] and the spectral equivalence was the main principle in the renormalization of an heterogeneous fault [3].

In Section 2 we formulate the evolution problem for the wave equation with nonlinear boundary conditions. This is modelling the antiplane shearing on a periodic system of collinear faults under a slip-dependent friction in an homogeneous linear elastic domain. The boundary conditions are linearized about an equilibrium state to obtain a linear initial and boundary-value problem. Then the associated spectral problem is considered. In order to capture the global behavior of the spectrum we shall distinguish between the *geometrical period* of the fault system and the *physical period* of the corresponding spectral solution and we shall seek for eigenfunctions which have the (physical) periodicity a multiple of the (geometrical) periodicity of the faults.

The dynamic spectral problem is studied in Section 3 for such an arbitrary periodicity. We are looking for the structure of the spectrum and the dependence of the first eigenvalue on a non-dimensional friction parameter. We prove that the spectrum consists of a decreasing and unbounded sequence of eigenvalues. They are shown to be increasing functions of the friction parameter.

Section 4 deals with the analysis of a static eigenproblem. Its solutions represent the critical values of the friction parameter for which the dynamical eigenvalues are vanishing. They characterize the existence of unstable solutions for the initial and boundary-value friction problem. We prove that these static eigenvalues form an increasing sequence. Then the problem is reduced to a system of hyper-singular integral equations which is solved through a semi-analytical technique. Numerical solutions for this system are given in Section 5. The computation results enable us to conjecture that the first eigenvalues/eigenfunctions of different physical periodicity are all equal and that the second eigenvalues converge to this first common eigenvalue as their physical period becomes indefinitely large.

2 Physical model

Consider the anti-plane shearing of a homogeneous linear elastic space containing a system of faults Γ_f, situated in the plane $y = 0$ and on which a slip-dependent friction law is supposed. We assume that the displacement field is 0 in directions Ox, Oy and that u_z does not depend on z. The displacement is therefore simply denoted by $w(t, x, y)$. The elastic media have the shear rigidity G, the density ρ

and the shear velocity $c = \sqrt{G/\rho}$. The non-vanishing shear stress components are $\sigma_{zx} = \tau_x^\infty + \frac{G}{L}\frac{\partial w}{\partial x}(t,x,y), \sigma_{zy} = \tau_y^\infty + \frac{G}{L}\frac{\partial w}{\partial y}(t,x,y)$ and the normal stress on the fault plane is $\sigma_{yy} = -S$ (S =const>0). Here L is a characteristic length of the problem and x, y are the corresponding non-dimensional space variables. The equation of motion is :

$$\frac{\partial^2 w}{\partial t^2}(t,x,y) = \frac{c^2}{L^2}\nabla^2 w(t,x,y) \tag{1}$$

for $t > 0, y \neq 0$. The boundary conditions on fault plane Γ_f are :

$$\begin{aligned}
\sigma_{zy}(t,x,0+) &= \sigma_{zy}(t,x,0-), && (2)\\
\sigma_{zy}(t,x,0) &= \mu(x,[w](t,x))S\ sign(\ \frac{\partial [w]}{\partial t}(t,x)) \ \text{ if } \ \frac{\partial [w]}{\partial t}(t,x) \neq 0, && (3)\\
|\sigma_{zy}(t,x,0)| &\leq \mu([w](t,x))S \quad \text{if} \quad \frac{\partial [w]}{\partial t}(t,x) = 0, && (4)
\end{aligned}$$

where $[w](t,x) = w(t,x,0+) - w(t,x,0-)$ is the relative slip.

The initial conditions are denoted by w_0 and w_1, i.e. :

$$w(0,x,y) = w_0(x,y), \quad \frac{\partial w}{\partial t}(0,x,y) = w_1(x,y). \tag{5}$$

For simplicity, let us assume in the following that the slip $[w]$ and the slip rate $\partial_t[w]$ are positive and the friction law is homogeneous on the fault plane having the form of a piecewise linear function:

$$\mu(x,u) = \mu_s - \frac{\mu_s - \mu_d}{2D_c}u \ \text{ if } \ u \leq D_c, \ \ \mu(x,u) = \mu_d \ \text{ if } \ u > D_c, \tag{6}$$

where u is the relative slip, μ_s and μ_d ($\mu_s > \mu_d$) are the static and dynamic friction coefficients, and D_c is the critical slip. This piecewise linear function is a reasonable approximation of the experimental observations reported by [16].

Since our intention is to study the evolution of the elastic system near an unstable equilibrium position, we shall suppose that $\tau_y^\infty = S\mu_s$. We remark that taking w as a constant satisfies (1)-(4) hence $w \equiv 0$ is an equilibrium position. For symmetry reasons, we may put $w(t,x,y) = -w(t,x,-y)$, hence we consider only one half-space $y > 0$ in (1),(5). With these assumptions (2)-(4) become :

$$\begin{aligned}
w(t,x,0+) &= 0 \quad x \notin \Gamma_f && (7)\\
\frac{\partial w}{\partial y}(t,x,0+) &= -\beta w(t,x,0+), \quad x \in \Gamma_f && (8)
\end{aligned}$$

if $w(t,x,0+) \leq D_c$ and for $w(t,x,0+) > D_c$ we have

$$\frac{\partial w}{\partial y}(t,x,0+) = -\beta D_c, \quad x \in \Gamma_f \tag{9}$$

where β is a non-dimensional parameter which will play an important role in our further analysis and it is given by

$$\beta = \frac{(\mu_s - \mu_d)SL}{GD_c}.$$

Since the initial perturbation (w_0, w_1) of the equilibrium $(w \equiv 0)$ is small we have $w(t, x, 0+) \leq D_c$ for $t \in [0, T_c]$ for all x, where T_c is a critical time for which the slip on the fault reaches the critical value D_c at least at one point, i.e. $\sup_{x \in R} w(T_c, x, 0+) = D_c$. Hence for a first period $[0, T_c]$ we deal with a linear initial and boundary value problem (1),(5),(7),(8). Our aim is to analyze the evolution of the perturbation during this initial phase.

Let us consider the following eigenvalue problem (connected to (1),(5),(7),(8)): find a bounded eigenfunction $\Phi : R \times R_+ \to R$ and the eigenvalue λ^2 such that

$$\nabla^2\Phi(x, y) = \lambda^2\Phi(x, y), \quad y > 0, \tag{10}$$

$$\frac{\partial\Phi}{\partial y}(x, 0+) = -\beta\Phi(x, 0+), \quad x \in \Gamma_f \tag{11}$$

$$\Phi(x, 0+) = 0, \quad x \notin \Gamma_f. \tag{12}$$

The eigenvalues λ^2 are functions of the parameter β, i.e. $\lambda^2 = \lambda^2(\beta)$. It is important to obtain a simple condition on β to determine the positiveness of the eigenvalues λ^2, representing an unstable behavior for the solution of the dynamic problem. The domain of existence of unstable solutions (i.e. $\lambda^2 > 0$) is limited by the critical values of the parameter β, for which $\lambda^2(\beta) = 0$. For this reason we transform (13)-(15) into a new eigenvalue problem corresponding to the static case (i.e. $\lambda^2 = 0$), in which the unknown are $\beta \geq 0$ and the eigenfunction $\varphi : R \times R_+ \to R$

$$\nabla^2\varphi(x, y) = 0, \quad y > 0, \tag{13}$$

$$\frac{\partial\varphi}{\partial y}(x, 0+) = -\beta\varphi(x, 0+), \quad x \in \Gamma_f \tag{14}$$

$$\varphi(x, 0+) = 0, \quad x \notin \Gamma_f. \tag{15}$$

The fault system Γ_f will be supposed to have a periodic geometry. In order to capture the global behavior of the spectrum we shall distinguish between the *geometrical period* of the fault system and the *physical period* of the corresponding spectral solution. In the next two sections we shall look for periodic eigenfunctions on domain which contains an arbitrary distributed system of faults. After that these eigenfunctions will be used to determine the solutions of the spectral problem. Specifically we seek for eigenfunctions which have the (physical) periodicity a multiple of the (geometrical) periodicity of the faults.

3 The dynamic spectral problem

Consider N arbitrary faults and denote by F their reunion. We suppose that $F \subset [0, T], T > 0$, with $F = \cup_{i=1}^{N}[a_i, b_i]$ and that we have an infinite set of faults Γ_f, in which the geometry of the finite system F is repeated periodically. Namely

$$\Gamma_f = \cup_{k\in\mathbb{Z}} F_k, \quad F_k = \cup_{i=1}^{N}[a_i + kT, b_i + kT] \tag{16}$$

Let $\Omega = (0,T)\times(0,+\infty)$. Since Φ is a periodic function with respect to x, we can consider it only on Ω and the periodic eigenvalue problem consists in finding a eigenfunction $\Phi : \Omega \to R$ and the eigenvalue λ^2 such that

$$\nabla^2\Phi = \lambda^2\Phi, \quad \text{in } \Omega, \tag{17}$$

$$\frac{\partial\Phi}{\partial y}(x,0+) = -\beta\Phi(x,0+), \quad x\in F \quad \Phi(x,0+) = 0, \quad x \notin F, \tag{18}$$

$$\Phi(0+,y) = \Phi(T-,y), \quad \frac{\partial\Phi}{\partial x}(0+,y) = \frac{\partial\Phi}{\partial x}(T-,y), \quad y>0. \tag{19}$$

We introduce the functional space of finite elastic and kinetic energy W

$$W = \{v \in H^1(\Omega)/v(x,0+) = 0 \;\; x\notin F, \quad v(0+,y) = v(T-,y) \;\; y>0\}$$

to deduce the variational formulation of the above eigenvalue problem

$$\Phi \in W, \quad \int_\Omega \nabla\Phi\cdot\nabla v \; dxdy + \lambda^2\int_\Omega \Phi v \; dxdy = \beta\int_F \Phi v \; dx, \quad \forall v\in W. \tag{20}$$

We shall restrict ourselves to the case of positive eigenvalues λ^2 which have important physical significance. Indeed as we have already explained in the previous subsection, the unstable evolution of a perturbation during the initiation phase can be described by the dominant part which is constructed with the eigenfunctions corresponding to positive eigenvalues.

Theorem 3.1 *There exists a sequence $(B_n(s),\Phi_n(s))_{n\in\mathbb{N}}$ of couples of functions $B_n : (0,+\infty)\to \mathbb{R}_+$ and $\Phi_n : (0,+\infty)\to V$ with the following property : (λ^2,Φ) is a solution of (17)-(19) with $\lambda^2>0$ if and only if there exists $n\in\mathbb{N}$ such that $\beta = B_n(\lambda^2)$ and $\Phi = \Phi_n(\lambda^2)$.*

Proof. Let $s>0$ be fixed and let us denote by $T : L^2(F)\to W$ the linear and bounded operator which associates to each $f\in L^2(F)$ the unique solution $T(f)\in W$ of the following linear equation

$$\int_\Omega \nabla T(f)\cdot\nabla v \; dxdy + s\int_\Omega T(f)v \; dxdy = \int_F fv \; dx, \quad \forall v\in W. \tag{21}$$

We can define now the linear bounded operator $K : L^2(F)\to H^{\frac{1}{2}}(F)$ by $K(f) = T(f)$ on F. From (38) we get

$$\int_F K(f)g \; dx = \int_\Omega \nabla T(f)\cdot\nabla T(g) \; dxdy + s\int_\Omega T(f)T(g) \; dxdy = \int_F fK(g) \; dx, \tag{22}$$

for all $f,g\in L^2(F)$, which implies that K is symmetric and positively defined. From the compact imbedding of $H^{\frac{1}{2}}(F)\subset L^2(F)$ we deduce that $K : L^2(F)\to L^2(F)$ is compact. Let us remark that if $K(f)=0$ then $T(f)=0$. That is, the kernel of K is 0. This yields the existence of $(\delta_n, h_n)_{n\in\mathbb{N}} \subset \mathbf{R}\times L^2(F)$, the sequence of couples eigenvalues/eigenfunctions for K, with $\lim\limits_{n\to\infty}\delta_n = 0$ and δ_n real and positive. A rearrangement of terms enables us to get δ_n as a decreasing sequence, with δ_0 the upper-value. Then we have

$$\delta_0 \int_F \eta K(f) f \, dx \geq \int_F \eta (K(f))^2 \, dx, \quad \forall f \in L^2(F). \tag{23}$$

If we define $(B_n(s), \Phi_n(s))$ as $B_n(s) =: \dfrac{1}{\delta_n}$ and $\Phi_n(s) =: T(h_n)$ then from $K(h_n) = \delta_n h_n$ and (21) we have

$$\int_\Omega \nabla\Phi_n(s)\cdot\nabla v \, dxdy + s\int_\Omega \Phi_n(s) v \, dxdy = B_n(s)\int_F \Phi_n(s) v \, dx, \tag{24}$$

for all $v \in W$. We have just proved that if $\beta = B_n(\lambda^2)$ then $(\lambda^2, \Phi_n(\lambda^2))$ is a solution for (20).

Conversely let (λ^2, Φ) be a solution for (20) for some $\lambda^2 > 0$. Let us denote by $s = \lambda^2$, then we put h, the trace of Φ on F, into (20) and (21) to deduce that $\Phi = \beta T(h)$, i.e. $\beta K(h) = h$. Hence there exists $n \geq 0$ such that $h = h_n$ and $\beta = \dfrac{1}{\delta_n}$ which imply that $\beta = B_n(\lambda^2)$ and $\Phi = \Phi_n(\lambda^2)$. $\Box$

Theorem 3.2 *i) The sequence* $(B_n(s))_{n\in\mathbb{N}}$ *satisfies*

$$0 < B_0(s) \leq B_1(s) \leq \cdots \leq B_n(s) \leq B_{n+1}(s) \leq \cdots, \quad \forall s \in (0, +\infty), \tag{25}$$

$$\lim_{n\to+\infty} B_n(s) = +\infty, \quad \forall s > 0. \tag{26}$$

$$\liminf_{s\to+\infty} \frac{B_n(s)}{\sqrt{s}} \geq \bar{C} > 0, \quad \forall n \geq 0, \tag{27}$$

ii) $s \to B_n(s)$ *is an increasing concave function and we can define*

$$\beta_n =: \lim_{s\to 0+} B_n(s),$$

iii) For all $\beta < \beta_0$ *there exist no solution of 17)-(19) with positive eigenvalues* $\lambda^2 > 0$*. If* $\beta \in (\beta_n, \beta_{n+1})$ *then 17)-(19) has n positive eigenvalues.*

iv) The functions $\lambda_n^2(\beta) =: B_n^{-1}(\beta)$ *are single-valued, convex and increasing on* $(\beta_n, +\infty)$*.*

Proof. We use the same notations as in the proof of the above theorem

i) Having in mind that δ_n is a decreasing and positive sequence with $\lim\limits_{n\to\infty}\delta_n = 0$ and since $B_n(s) =: \dfrac{1}{\delta_n}$ we deduce (25)-(26). Let us prove now (27). If we put $w = \Phi_n(s)$ in the above equation we obtain the following inequality for $s > 1$:

$$||\Phi_n(s)||^2_{H^1(\Omega)} + (s-1)||\Phi_n(s)||^2_{L^2(\Omega)} \leq B_n(s)||\Phi_n(s)||^2_{L^2(F)} \tag{28}$$

We use now the following inequality (see [?], Lemma 5.1 for a simple proof)

$$||v||^2_{L^2(F)} \leq C||v||_{L^2(\Omega)}||v||_{H^1(\Omega)}, \quad \forall v \in V, \tag{29}$$

to get $CB_n(s) \geq 1/q_n(s)+(s-1)q_n(s) \geq 2\sqrt{s-1}$ with $q_n(s) =: ||\Phi_n(s)||_{L^2(\Omega)}/||\Phi_n(s)||_{H^1(\Omega)}$ and (27) follows.

ii) We denote by

$$L_n = \overline{Sp\{h_n, h_{n+1}, ...\}}, \quad S_n = \{f \in L_n; \int_F f^2\, dx = 1\},$$

where the closure is in $L^2(\Gamma_f)$. We remark that $B_n = 1/\delta_n$, as a function of s, is the lower bound of a family of affine functions

$$B_n(s) = \frac{1}{\delta_n} = \inf_{f\in S_n} \int_{\Gamma_f} \eta K(f) f\, dx = \inf_{f\in S_n} \int_\Omega |\nabla T(f)|^2\, dxdy + s\int_\Omega |T(f)|^2\, dxdy, \tag{30}$$

hence B_n it is a concave function. This property and the fact that $\liminf_{s\to+\infty} B_n(s) = +\infty$ (see ii)) implies that B_n is increasing.

iii) is a direct consequence of the previous theorem and ii).

iv) is a direct consequence of ii) and iii). $\square$

4 The static spectral problem

Since the intersection points β_n of the dynamic eigenvalue $\lambda^2(\beta)$ with the axis $\lambda^2 = 0$ provide stability properties of the dynamic solution, we here restrict our analysis to the static eigenvalue problem. Due to the absence of kinetic energy, the solution space will change requiring a specific analysis.

Considering the same geometry and notations as in the previous section the static periodic eigenvalue problem consists in finding $\varphi : \Omega \to R$ and β such that

$$\nabla^2\varphi = 0, \quad \text{in } \Omega \tag{31}$$

$$\varphi(x, 0+) = 0, \quad x \notin F, \quad \frac{\partial\varphi}{\partial y}(x, 0+) = -\beta\varphi(x, 0+), \quad x \in F \tag{32}$$

$$\varphi(0+, y) = \varphi(T-, y), \quad \frac{\partial\varphi}{\partial x}(0+, y) = \frac{\partial\varphi}{\partial x}(T-, y), \quad y > 0. \tag{33}$$

We introduce, as in [14], the functional space of finite elastic energy V. Let $\mathcal{V}$ be the following subspace of $H^1(\Omega)$

$$\mathcal{V} = \{v \in H^1(\Omega); v(0, y) = v(T, y),\ y > 0, \quad v(x, 0) = 0,\ x \notin F, \\ v = 0 \text{ in a neighborhood of } y = +\infty\} \tag{34}$$

endowed with the norm $|| \, ||_V$ generated by the following scalar product

$$(u,v)_V = \int_\Omega \nabla u \cdot \nabla v dx dy, \quad ||u||_V^2 = (u,u)_V, \quad \forall u,v \in \mathcal{V}. \tag{35}$$

We define V as the closure of $\mathcal{V}$ in the norm $||u||_V$. The space V is continuously embedded in $H^1(\Omega_R)$ for all $R > 0$, with $\Omega_R = (0,T) \times (0,R)$, but V is not a subspace of $H^1(\Omega)$. Indeed, if $v \in V$ then v is not vanishing for $y \to +\infty$. Actually, one can prove that for φ a solution of (31)-(33) we have

$$\lim_{y \to +\infty} \varphi(x,y) = \frac{1}{T} \int_F \varphi(s,0) ds, \quad \forall x \in F$$

The eigenvalue problem (31)-(33) has following variational formulation

$$\varphi \in V, \quad (\varphi, v)_V = \beta \int_F \varphi(x,0) v(x,0) \, dx, \quad \forall \, v \in V. \tag{36}$$

The following theorem, which is the main result of this section, gives the structure of the spectrum.

Theorem 4.1 *The eigenvalues and eigenfunctions of (36) consists of a sequence $(\beta_n, \varphi_n)_{n \in \mathbb{N}}$ of with $0 < \beta_0 \leq \beta_1 \leq .....$ and $\beta_n \longrightarrow +\infty$. Moreover we have*

$$||v||_V^2 \geq \beta_0 \int_F v^2 \, dx, \quad \forall v \in V. \tag{37}$$

Proof. Let us denote by $T : L^2(F) \to V$ the linear and bounded operator which associates to each $f \in L^2(F)$ the unique solution $T(f) \in V$ of the following linear equation

$$\int_\Omega (T(f), v)_V = \int_F f v \, dx, \quad \forall v \in V. \tag{38}$$

We can define now the linear bounded operator $K : L^2(F) \to H^{\frac{1}{2}}(F)$ by $K(f) = T(f)$ on F. From (38) we get

$$\int_F K(f) g \, dx = (T(f), T(g))_V = \int_F f K(g) \, dx, \tag{39}$$

for all $f, g \in L^2(F)$, which implies that K is symmetric and and positively defined. From the compact imbedding of $H^{\frac{1}{2}}(F) \subset L^2(F)$ we deduce that $K : L^2(F) \to L^2(F)$ is compact. Let us remark that if $K(f) = 0$ then $T(f) = 0$. That is, the kernel of K is 0. This yields the existence of $(\delta_n, h_n)_{n \in \mathbb{N}} \subset \mathbf{R} \times L^2(F)$, the sequence of couples eigenvalues/eigenfunctions for K, with $\lim_{n \to \infty} \delta_n = 0$ and δ_n real and positive. A rearrangement of terms enables us to get δ_n as a decreasing sequence, with δ_0 the upper-value. Then we have

$$\delta_0 \int_F K(f) f \, dx \geq \int_F (K(f))^2 \, dx, \quad \forall f \in L^2(F). \tag{40}$$

If we define (β_n, φ_n) as $\beta_n =: \dfrac{1}{\delta_n}$ and $\Phi_n =: T(h_n)$ then (36) holds and from (40) we deduce (37). $\square$

Here we give a solution method for the periodic eigenvalue problem (31)-(33). We first reduce it to a system of hypersingular integral equations and then we use an appropriate semi-analytic method to find solutions of period T.

The Fourier transform in x of the equation (14) leads to

$$\frac{\partial^2 \widehat{\varphi}}{\partial y^2} = \xi^2 \widehat{\varphi} \tag{41}$$

where $\widehat{\varphi}(\xi, y)$ is the Fourier transform in x of $\varphi(x, y)$. The finite-energy solutions of (41), which require that $\nabla\varphi$ is vanishing at infinity, have the form:

$$\widehat{\varphi}(\xi, y) = A(\xi)e^{-|\xi|y}. \tag{42}$$

The Fourier inverse of (42) is

$$\varphi(x, y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} A(\xi)e^{-|\xi|y - i\xi x}d\xi \tag{43}$$

and for $y = 0$ it leads to

$$A(\xi) = \int_{-\infty}^{+\infty} \varphi(s, 0)e^{i\xi s}ds \tag{44}$$

By substitution of $A(\xi)$ in (43) and interchange of the integration order we get

$$\varphi(x, y) = \frac{y}{\pi}\int_{\Gamma_f} \frac{\varphi(s, 0)}{y^2 + (s - x)^2}ds \tag{45}$$

which is a representation formula for the displacement field $\varphi(x, y)$. To deduce (45) we have used the relation $\displaystyle\int_0^{+\infty} e^{-\xi y}\cos(\xi(s - x))d\xi = \frac{y}{y^2 + (s - x)^2}$ (see [9], ch. 1.4). By derivation with respect to y we obtain

$$\frac{\partial\varphi}{\partial y}(x, y) = \frac{1}{\pi}\int_{\Gamma_f} \varphi(s, 0)\frac{(s - x)^2 - y^2}{(y^2 + (s - x)^2)^2}ds \tag{46}$$

which, for $y = 0$, gives

$$\frac{\partial\varphi}{\partial y}(x, 0) = \frac{1}{\pi}FP\int_{\Gamma_f} \frac{\varphi(s, 0)}{(s - x)^2}ds \tag{47}$$

where the integral is taken in the finite part sense.

For $x \in F$, from the boundary condition (32), we have

$$\beta\varphi(x, 0) = -\frac{1}{\pi}FP\int_{\Gamma_f} \frac{\varphi(s, 0)}{(s - x)^2}ds \tag{48}$$

which is a hyper-singular integral equation for $\varphi(x,0)$.

Then, making use of (33), the integral equation (48) becomes

$$\begin{aligned}\beta\varphi(x,0) &= -\frac{1}{\pi}\sum_{k\in\mathbb{Z}} FP\int_{F_k}\frac{\varphi(s,0)}{(s-x)^2}ds \\ &= -\frac{1}{\pi}\sum_{k\in\mathbb{Z}} FP\int_{F}\frac{\varphi(u,0)}{(u+kT-x)^2}du \end{aligned} \tag{49}$$

Interchanging the series with the integral and using the formula [10]:

$$\sum_{k\in\mathbf{Z}}\frac{1}{(u+kT-x)^2} = \frac{\pi^2}{T^2}\csc^2(\frac{\pi}{T}(x-u)) \tag{50}$$

the equation (49) modifies as

$$\beta\varphi(x,0) = -\frac{\pi}{T^2}FP\int_F \varphi(u,0)\csc^2(\frac{\pi}{T}(x-u))du \tag{51}$$

for $x\in P$.

Since (51) contains no more infinite series, this equation is now suitable for a numerical integration. For an efficient solving, we further look for a particular development of the spectral solution which takes into account the boundary conditions at the faults endpoints.

Introducing $\varphi_i : [a_i, b_i] \to \mathbb{R}$ through

$$\varphi_i(t) = \varphi(\frac{b_i-a_i}{2}t + \frac{a_i+b_i}{2}, 0) \tag{52}$$

and the transformation variables s and t:

$$x = t\frac{b_k-a_k}{2} + \frac{a_k+b_k}{2} \qquad u = s\frac{b_i-a_i}{2} + \frac{a_i+b_i}{2} \tag{53}$$

we can write (51) as the system:

$$\begin{aligned}\beta\varphi_k(t) &= -\frac{\pi}{T^2}\sum_{i=1}^{N}(\frac{b_i-a_i}{2}FP\int_{-1}^{1}\varphi_i(s)\csc^2(\frac{\pi}{T}(\frac{b_k-a_k}{2}t+\frac{a_k+b_k}{2} \\ &\quad -\frac{b_i-a_i}{2}s-\frac{a_i+b_i}{2}))ds)\end{aligned} \tag{54}$$

for $t\in[-1,1]$.

The finite-part integral may give rise to difficulties in performing numerical integrations of this system. For this reason we shall separate the singular terms in (54) and calculate them analytically. Let us introduce the function

$$g(z) = \begin{cases} \csc^2(z) - \frac{1}{z^2} & for\ z\neq 0 \\ \frac{1}{3} & for\ z = 0 \end{cases} \tag{55}$$

Since we have the development

$$\csc^2(z) = \frac{1}{z^2} + \frac{1}{3} + O(z^2) \tag{56}$$

we can decompose $\csc^2(z)$ as the sum of the singular term $\dfrac{1}{z^2}$ and the function $g(z)$:

$$\csc^2(z) = \frac{1}{z^2} + g(z) \tag{57}$$

Using this decomposition in (54) we get the following system of integral equations

$$\beta\varphi_k(t) = -\frac{2}{\pi(b_k - a_k)} FP \int_{-1}^{1} \frac{\Phi_k(s)}{(s-t)^2} ds - \frac{\pi}{T^2} \sum_{i=1}^{N} \int_{-1}^{1} H_{ik}(t,s)\varphi_i(s) ds \tag{58}$$

for $t \in (-1,1)$ and where

$$H_{ik}(t,s) = \frac{b_k - a_k}{2} g\left(\frac{\pi(s-t)}{T}\frac{b_k - a_k}{2}\right)\delta_{ik}$$
$$+\frac{b_i - a_i}{2}\csc^2\left(\frac{\pi}{T}\left(t\frac{b_k - a_k}{2} + \frac{b_k + a_k}{2} - s\frac{b_i - a_i}{2} - \frac{b_i + a_i}{2}\right)\right)(1 - \delta_{ik}) \tag{59}$$

We look for the solution of this system in the form of the expansion

$$\varphi_k(t) = \sum_{n=1}^{\infty} U_{nk} \sin(n \arccos(t)) \tag{60}$$

on $[-1,1]$. Replacement of this expression in (58) and use of the changes of variables $t = \cos\theta, s = \cos\psi$ for $\theta, \psi \in [0,\pi]$ and the Glauert integral formula

$$\int_0^{\pi} \frac{\cos(n\psi)d\psi}{\cos\psi - \cos\theta} = \pi\frac{\sin(n\theta)}{\sin\theta} \tag{61}$$

yields

$$\beta\sum_{n=1}^{\infty} U_{nk}\sin(n\theta) = \frac{2}{b_k - a_k}\sum_{n=1}^{\infty} U_{nk}\frac{n\sin(n\theta))}{\sin\theta}$$
$$-\frac{\pi}{T^2}\sum_{i=1}^{N}\sum_{n=1}^{\infty} U_{ni}\int_0^{\pi}\sin(n\psi))H_{ik}(\cos\theta,\cos\psi)\sin\psi d\psi \tag{62}$$

for $k = \overline{1,N}$. Let us now multiply these equations by $\sin\theta\sin(p\theta)$ and integrate on $[0,\pi]$. It results

$$\beta\sum_{n=1}^{\infty} U_{nk}\int_0^{\pi}\sin(n\theta)\sin\theta\sin(p\theta)d\theta \tag{63}$$
$$= \frac{2}{b_k - a_k}\sum_{n=1}^{\infty} nU_{nk}\int_0^{\pi}\sin(n\theta)\sin(p\theta)d\theta$$
$$-\frac{\pi}{T^2}\sum_{i=1}^{N}\sum_{n=1}^{\infty} U_{ni}\int_0^{\pi}\int_0^{\pi}\sin(n\psi)\sin\psi\sin(p\theta)\sin\theta H_{ik}(\cos\theta,\cos\psi)d\psi d\theta \tag{64}$$

In order to simplify this system we introduce the notations

$$A_{pn} \equiv \int_0^\pi \sin(n\theta)\sin\theta\sin(p\theta)d\theta \tag{65}$$

$$B_{pn} \equiv n\int_0^\pi \sin(n\theta)\sin(p\theta)d\theta \tag{66}$$

$$C_{pn}^{ki} \equiv -\int_0^\pi\int_0^\pi \sin(n\psi)\sin(p\theta)\sin\psi\sin\theta H_{ik}(\cos\theta,\cos\psi)d\psi d\theta \tag{67}$$

The first two expressions can be explicitly calculated to obtain:

$$A_{mn} = -\frac{2mn(1+(-1)^{m+n})}{((m-n)^2-1)((m+n)^2-1)}(1-\delta_{n,m-1})(1-\delta_{n,m+1}) \tag{68}$$

$$B_{mn} = \frac{n\pi}{2}\delta_{m,n} \tag{69}$$

Since the integrant in the last formula (67) is a continuous function a numerical integration can be used to evaluate C_{pn}^{mki}.

With these notations the system of integral equations can be written in a compact form:

$$\beta\frac{b_k-a_k}{2}\sum_{n=1}^{\infty}A_{pn}U_{nk} = \sum_{i=1}^{N}\sum_{n=1}^{\infty}D_{pn}^{ki}U_{ni} \tag{70}$$

with

$$D_{pn}^{ki} \equiv \begin{cases} \frac{b_k-a_k}{2}\frac{\pi}{T^2}C_{pn}^{ki}+B_{pn} & for\ i=k \\ \frac{b_k-a_k}{2}\frac{\pi}{T^2}C_{pn}^{kk} & for\ i\neq k \end{cases} \tag{71}$$

Relation (70) is a generalized eigenvalue problem. In order to compute the corresponding matrices, we shall truncate the infinite series up to N. The eigenvalue form appears more clearly by defining the $N\times N$ matrices

$$\mathcal{A}_{lr}^N \equiv \frac{b_k-a_k}{2}A_{pn}\delta_{i,k} \quad ; \quad \mathcal{D}_{lr}^N \equiv D_{pn}^{ki} \tag{72}$$

with $l = N\times(k-1)+p$ and $r = N\times(i-1)+n$ for $p,n=\overline{1,N}$ and $i,k=\overline{1,N}$. Also the generalized eigenvectors

$$v_r = U_{ni} \tag{73}$$

for $r = N\times(i-1)+n$ with $n=\overline{1,N}$ and $i=\overline{1,N}$.

With (72) and (73) the system (70) now reduces to the generalized eigen-problem

$$\beta\sum_{q=1}^{N\times N}\mathcal{A}_{lq}^N v_q = \sum_{q=1}^{N\times N}\mathcal{D}_{lq}^N v_q \tag{74}$$

for the eigenvalue β and the eigenvectors v_q.

5 Numerical results

We consider hereafter a periodic system of faults Γ_f. Let $P > 0$ be the geometrical period and let $[a, b] \subset [0, P]$ be the reference fault. Namely

$$\Gamma_f = \cup_{k\in\mathbb{Z}}\Gamma_k, \quad \Gamma_k = [a + kP, b + kP] \tag{75}$$

Due to the periodicity of the problem (13)-(15), we expect a periodic behavior for the eigenfunction $x \to \varphi(x, y)$. To better point out this periodicity of the solutions, let us distinguish between the *geometrical period* of the fault system and the *physical period* of the corresponding spectral solution. Specifically, for a period P, we shall seek for eigenfunctions $\varphi^m(x, y)$ of the static eigenvalue problem of periodicity mP with $m = 1, 2, ...$ That is, we are looking for $\varphi^m : R \times R_+ \to R$ and β^m solution of

$$\nabla^2\varphi^m(x, y) = 0, \quad y > 0, \tag{76}$$

$$\varphi^m(x + mP, y) = \varphi^m(x, y) \tag{77}$$

$$\frac{\partial\varphi^m}{\partial y}(x, 0+) = -\beta^m\varphi^m(x, 0+), \quad x \in \Gamma_f, \tag{78}$$

$$\varphi^m(x, 0+) = 0, \quad x \notin \Gamma_f. \tag{79}$$

We remark that (76)-(79) can be obtained form (31)-(33) with the parameter set: $T = mP$, $N = m$, $a_i = a + iP$ and $b_i = b + iP$, with $i = \overline{1, m}$ and $\beta = \beta$. Hence for each m we use the results of section 4.

As it follows from Theorem 1 for all m, the eigenvalues are an unbounded non-decreasing sequence, i.e.

$$0 < \beta_0^m \leq \beta_1^m \leq ... \leq \beta_n^m \leq ..., \qquad \lim_{n\to\infty} \beta_n^m = +\infty. \tag{80}$$

The system (76)-(79) was numerically solved for different values of m. We obtained that the first eigenvalue/eigenfunction β_0^1/φ_0^1 of the elementary periodicity P is also the first eigenvalue/eigenfunction of periodicity mP for all m, simply denoted by β_0/φ_0 i.e.

$$\beta_0^m = \beta_0^1 = \beta_0, \quad \text{for all} \quad m \geq 1$$

$$\varphi_0^m(x, y) = \varphi_0^1(x, y) = \varphi_0(x, y), \quad \text{for all} \quad m \geq 1.$$

The property that β_0 is the first eigenvalue for all periodicities seems to be a general feature for all the cases we have considered in our computations. This numerical result enable us to formulate

Conjecture. *For a geometric periodicity P the first eigenvalues/eigenfunctions of different (physical) periodicity mP are equal for all m.*

The system (76)-(79) was numerically solved using the method described in the previous section for $P = 4$ and $[a, b] = [1, 3]$. We have found that

$$\beta_0 = 1.03349339.$$

The first eigenvalue β_0 represents the threshold of instability, that is

$$w \equiv 0 \quad \text{is unstable when} \quad \beta > \beta_0 \tag{81}$$

Note that this value is inferior to the corresponding threshold of stability for a single fault computed in [5, 6] as $\beta_0 = 1.15777$.

In Figure 1 the computed eigen-function $\varphi_0(x,0)$ is plotted.

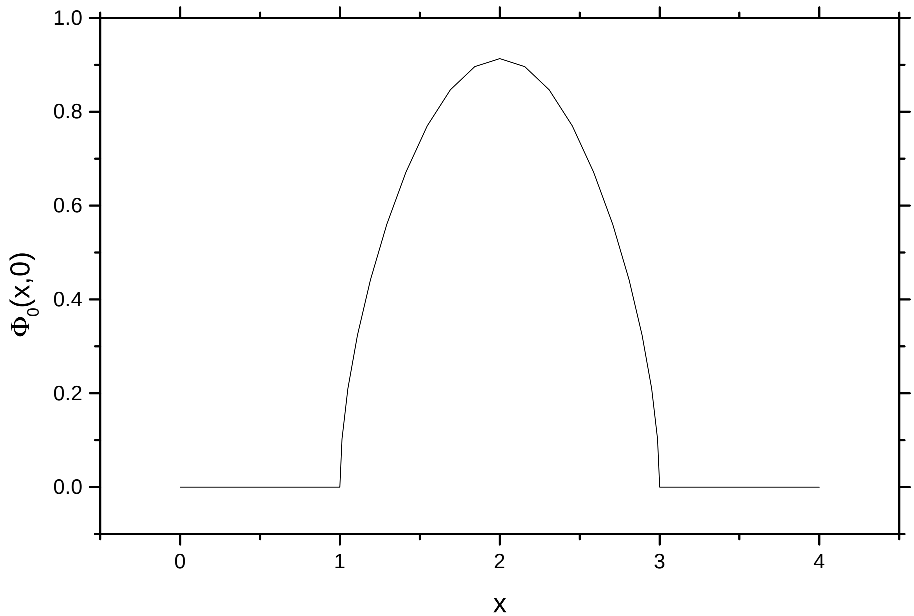

Figure 1: The first eigenfunction $\varphi_0(x,0)$ on a (geometrical) period.

The computed eigenvalues of β_1^n are presented in Table 1.

	$m=1$	$m=2$	$m=10$	$m=20$	$m=100$
β_1^n	2.764167	1.22235793	1.098298515	1.06730381	1.04040256

Table 1. The eigenvalue β_1^n for different values of m.

From the results in Table 1 one can formulate the

Conjecture. $\qquad \beta_1^n \to \beta_0 \quad \text{for} \quad n \to \infty.$

This shows that for $\beta > \beta_0$, arbitrarily close to β_0, em an infinite set of eigenfunctions φ_1^m, for the dynamic problem (13)-(15) have an unstable behavior ($\lambda^2(\beta) > 0$).

In Figure 2 the eigenfunction φ_1^{100} is plotted. We remark that the period is $100P$, by contrast to φ_0 of period P. An overall sinusoidal shape can be observed for the eigenfunction φ_1^{100}

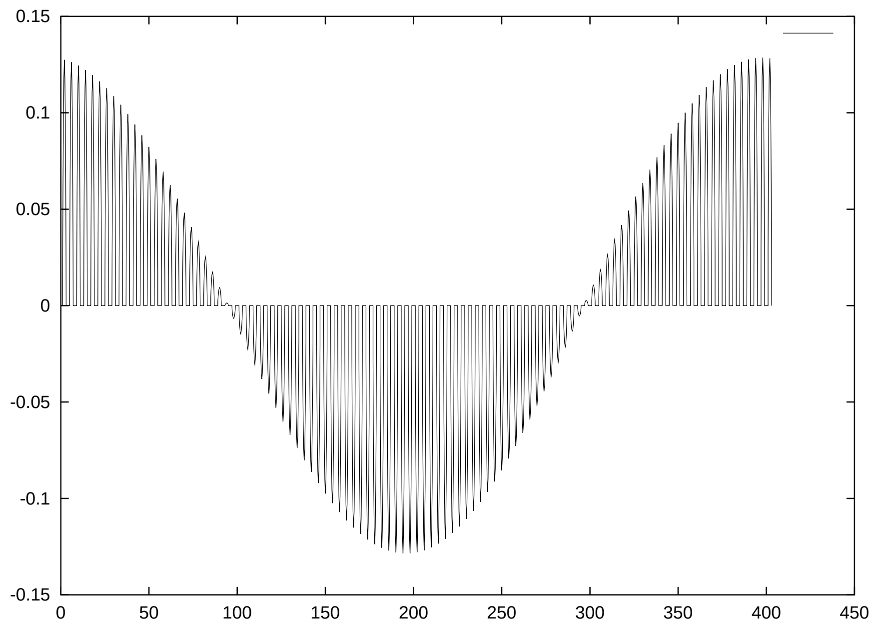

Figure 2: The computed eigenfunction $\varphi_1^{100}(x,0)$ versus the fault line axis x. Note the overall sinusoidal shape.

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