Dissipation scales of kinetic helicities in turbulence

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A systematic study of the influence of the viscous effect on both the spectra and the nonlinear fluxes of conserved as well as nonconserved quantities in Navier–Stokes turbulence is proposed. This analysis is used to estimate the helicity dissipation scale which is shown to coincide with the energy dissipation scale. However, it is shown using the decomposition of helicity into eigenmodes of the curl operator that viscous effects have to be taken into account for wave vectors smaller than the Kolmogorov wave number in the evolution of these eigencomponents of the helicity.
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I. INTRODUCTION

In two papers[1,2] based on a dimensional analysis and on the simulation of the Gledzer, Ohkitani, and Yamada (GOY) shell model,[3–5] it was suggested that dissipation of kinetic helicity occurs at a scale k_H^{-1} larger than the Kolmogorov scale k_E^{-1}. In contrast, using a different shell model of turbulence based on helical wave decomposition, both scales were found to be equal $k_H = k_E$.[6] In addition, direct numerical simulations, also presented in Ref. 6, seem to confirm the latter result though, as noted by the authors, the computational limitations prevent to have a Reynolds number sufficiently large to really discriminate between both scenarios.

The purposes of the present work are to investigate further the possible existence of a specific helicity dissipation scale and to understand why two shell models do exhibit different helicity behaviors while their energy spectra are very much similar. Part of this apparent contradiction comes from the very definition of the dissipation scale. Indeed, in the Kolmogorov theory, there is no ambiguity. The scale at which the energy dissipation terms are no longer negligible when compared to the nonlinear fluxes of energy corresponds to the scale at which the energy spectrum departs from the Kolmogorov power law. This scale marks the end of the cascade process, as well as the beginning of energy spectrum fall off.

The situation is less clear for nonconserved quantities such as the positive H^+ and the negative helicity H^- are defined, respectively, as the helicity carried on by the eigenvectors of the curl operator with positive and negative eigenvalues. Generally, for nonconserved quantities Q, we propose to refer to the dissipation scale as the scale after which the dissipative term *dominates* the dynamics, so that the spectrum of Q falls off. Such a scale might very well differ from the scale, referred hereafter as the *viscous scale*, at which the dissipative terms *start* to play a role in the dynamics of Q. Indeed, for a nonconserved quantity, the nonlinear term might very well compensate for the increase of dissipation in part of the high wave number range after the viscous scale and prevent the spectrum to fall off even if dissipation is active. In general, the viscous scale should be smaller than the dissipation scale. However, for conserved quantities, both the viscous and the dissipation scales coincide.

A general discussion on the determination of dissipation scale is presented in Sec. II for conserved as well as nonconserved quantities. The specific case of the two conserved quantities in three-dimensional turbulence, the energy, and the helicity is discussed in Sec. III. The positive and negative helicities, which are not conserved quantities, are discussed in Sec. IV. Shell models describing the high Reynolds number behavior of turbulence are discussed in Sec. V. Both models used in Refs. 2 and 6 are introduced and analyzed numerically in Sec. VI. It is shown very clearly that the dissipation scales for the helicity and the energy coincide and are given by the Kolmogorov length scale. Moreover, the dissipation scale for the positive and negative helicities also corresponds to the energy dissipation scale. However, the analysis of their fluxes allows to identify very clearly a viscous scale for both H^+ and H^- that is smaller than the energy dissipation scales.

II. DISSIPATION SCALES IN TURBULENT SYSTEMS WITH CASCADES

Before discussing the specific problem of energy or helicity dissipation scale, we consider a general quadratic quantity Q that is not necessarily conserved by the nonlinearities of the Navier–Stokes equation

$$Q = \int_V d^3\mathbf{r}\, a(\mathbf{r}) b(\mathbf{r}) = \int d^3\mathbf{k}\, \tilde{a}(\mathbf{k}) \tilde{b}(\mathbf{k})^* + \text{c.c.}. \qquad (1)$$

Here, $a(\mathbf{r})$ and $b(\mathbf{r})$ are two fields and $\tilde{a}(\mathbf{k})$ and $\tilde{b}(\mathbf{k})$ are their Fourier transforms. In the following, the system is assumed

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to be statistically isotropic. In that case, it is convenient to introduce the spectrum $Q(p)$, so that

$$Q = \int dpQ(p). \tag{2}$$

The parts of this quantity that are represented by modes such that $|\mathbf{k}|<\kappa$ and $|\mathbf{k}|>\kappa$ are denoted, respectively, by $Q^{<}(\kappa)$ and $Q^{>}(\kappa)$

$$Q^{<}(\kappa) = \int_{|\mathbf{k}|<\kappa} d^3\mathbf{k}\tilde{a}(\mathbf{k})\tilde{b}(\mathbf{k})^* + \text{c.c.}, \tag{3}$$

$$Q^{>}(\kappa) = \int_{|\mathbf{k}|>\kappa} d^3\mathbf{k}\tilde{a}(\mathbf{k})\tilde{b}(\mathbf{k})^* + \text{c.c.}, \tag{4}$$

$$Q = Q^{<}(\kappa) + Q^{>}(\kappa) \qquad \forall\ \kappa. \tag{5}$$

Their evolution is given by:

$$\partial_t Q^{<}(\kappa) = s_Q - \Pi_Q^{<}(\kappa) - d_Q^{<}(\kappa), \tag{6a}$$

$$\partial_t Q^{>}(\kappa) = -\Pi_Q^{>}(\kappa) - d_Q^{>}(\kappa), \tag{6b}$$

where s_Q is the source of Q here injected by a forcing process in the largest scales of the system (k_F) so that $k_F<\kappa$. In that case, the source term is independent of κ. The nonlinearity contributions to the evolution of $Q^{<}$ and $Q^{>}$ are noted, respectively, $\Pi_Q^{<}(\kappa)$ and $\Pi_Q^{>}(\kappa)$. They correspond to fluxes, respectively, outward and inward the sphere of radius κ if Q is a conserved quantity. The dissipation of Q in the modes $|\mathbf{k}|<\kappa$ ($|\mathbf{k}|>\kappa$) is noted $d_Q^{<}(\kappa)$ [$d_Q^{>}(\kappa)$]. In the following, the dissipative processes are assumed to be represented by viscous type terms, so that

$$d^{<}(\kappa)_Q = 2\nu\int_0^{\kappa} dpp^2Q(p), \tag{7a}$$

$$d^{>}(\kappa)_Q = 2\nu\int_{\kappa}^{\infty} dpp^2Q(p). \tag{7b}$$

If the system undergoes a cascading process that transfers Q from the forcing scales to small scales, the nonlinear transfer at scale κ should be characterized by a typical time scale that will be denoted $\tau_Q^{\text{nl}}(\kappa)$. On the other hand, dissipation processes should also be characterized by a time scale $\tau_Q^{\text{diss}}(\kappa)$. In the case of viscous type dissipation, $\tau_Q^{\text{diss}}(\kappa)=1/(\nu\kappa^2)$. The comparison of these characteristic time scales can be used to estimate the end of the cascade range (usually referred to as the inertial range as long as kinetic energy is concerned). Indeed, in the range dominated by the nonlinear interactions, $\tau_Q^{\text{nl}}(\kappa)<\tau_Q^{\text{diss}}(\kappa)$ since nonlinear interactions should be faster than dissipative processes. On the contrary, in the dissipation range, $\tau_Q^{\text{nl}}(\kappa)>\tau_Q^{\text{diss}}(\kappa)$. An estimate of the dissipation scale k_Q^D is thus

$$\tau_Q^{\text{nl}}(k_Q^D) \approx \tau_Q^{\text{diss}}(k_Q^D). \tag{8}$$

Of course, in order to predict k_Q^D, it is necessary to guess the expression for $\tau_Q^{\text{nl}}(\kappa)$. For instance, if a scaling law can be assumed [$\tau_Q^{\text{nl}}(\kappa)=A_Q\kappa^{-\alpha_Q}$], the dissipation wave number is given by

$$k_Q^D \propto \left(\frac{1}{\nu A_Q}\right)^{1/(2-\alpha_Q)}. \tag{9}$$

Another typical length scale can be introduced via Eq. (6a) and under the assumption that a stationary state can be reached

$$\Pi_Q^{<}(\kappa) = s_Q - 2\nu\int_0^{\kappa} dpp^2Q(p). \tag{10}$$

This expression can be used to obtain an estimate of the viscous scale k_Q^{ν}, at which the viscous term becomes important when compared to s_Q, by assuming that the spectrum $Q(p)$ follows a power law $Q(\kappa)=B_Q\kappa^{-\beta_Q}$

$$2\nu\int_0^{k_Q^{\nu}} dpp^2B_Qp^{-\beta_Q} \approx s_Q, \tag{11}$$

which leads to

$$k_Q^{\nu} \propto \left(\frac{s_Q}{\nu B_Q}\right)^{1/(3-\beta_Q)}. \tag{12}$$

For a conserved quantity, the spectrum has to fall off for $\kappa>k_Q^{\nu}$, otherwise the dissipation would exceed the injection rate and consequently the nonlinear term must vanish. It is thus expected that $k_Q^{\nu}=k_Q^D$. However, for a nonconserved quantity, the dissipation may exceed the injection rate since the nonlinear term does not necessarily vanish. Thus, nothing prevents the spectrum to remain $Q(\kappa)=B_Q\kappa^{-\beta_Q}$ for $\kappa>k_Q^{\nu}$ and this viscous scale k_Q^{ν} may be smaller than the dissipation scale k_Q^D.

III. ENERGY AND HELICITY DISSIPATION SCALES

We first consider the cascade of energy. The total energy injection rate is then usually noted $s_E=\epsilon$ and the Kolmogorov energy spectrum can be derived

$$E(k) = C_E\epsilon^{2/3}k^{-5/3} \tag{13}$$

in the inertial range. The estimate for the dissipation wave number based on the equality between the characteristic time scales requires an expression for $\tau_E^{\text{nl}}(\kappa)$. Various proposals can be found in the literature, but all yield the same scaling since they are built with only κ and ϵ, assuming the viscosity does not influence the nonlinear characteristic time

$$\tau_E^{\text{nl}}(\kappa) \propto \kappa^{-2/3}\epsilon^{-1/3}, \tag{14}$$

which means $A_E\propto\epsilon^{-1/3}$ and $\alpha_E=2/3$. Consequently, expression (9) yields

$$k_E^D \propto \left(\frac{1}{\nu\epsilon^{-1/3}}\right)^{3/4} \propto \left(\frac{\epsilon}{\nu^3}\right)^{1/4}. \tag{15}$$

Similarly, the Kolmogorov spectrum implies $\beta_E=5/3$ and $B_E\propto\epsilon^{2/3}$ and expression (12) yields the same estimate

$$k_E^{\nu} \propto \left(\frac{\epsilon}{\nu\epsilon^{2/3}}\right)^{3/4} \propto \left(\frac{\epsilon}{\nu^3}\right)^{1/4}. \tag{16}$$

We now consider the helicity cascade. Both studies presented in Refs. 1 and 6 make the assumption that the characteristic time of nonlinear transfer of energy and helicity are the same: $\tau_H^{\rm nl}(\kappa)=\tau_E^{\rm nl}(\kappa)$. Since both energy and helicity are dissipated by linear viscous processes, their dissipation characteristic time is obviously identical $[\tau_H^{\rm diss}(\kappa)=\tau_E^{\rm diss}(\kappa)=1/(\nu k^2)]$. In that case, the dissipation wave number for energy and helicity obtained by comparing the nonlinear transfer time to the dissipation time must coincide

$$k_H^D \propto k_E^D \propto \left(\frac{\epsilon}{\nu^3}\right)^{1/4}. \tag{17}$$

Also, the equality of the nonlinear transfer time is also known to imply the following helicity spectrum:

$$H(k) = C_H \delta \epsilon^{-1/3} k^{-5/3}, \tag{18}$$

where C_H is a dimensionless constant and δ is the helicity injection rate. In that case, formula (12) with $s_H=\delta$ and $B_H=\delta\epsilon^{-1/3}$ leads to the same expression

$$k_H^\nu \propto \left(\frac{\delta}{\nu\delta\epsilon^{-1/3}}\right)^{3/4} \propto \left(\frac{\epsilon}{\nu^3}\right)^{1/4}. \tag{19}$$

Hence, both approaches yield the same result and tend to confirm the equality between the helicity and the energy dissipation scales. However, although the equality of both dissipation scales is so obvious, the analysis becomes a bit more involved when using the helical decomposition of the energy and helicity spectra.

The dimensional analysis presented in Secs. II and III is not new, although the distinction between the viscous scale and the dissipation scale is not necessarily very common. Nevertheless, the introduction of these two scales is very important in the analysis of helicity cascades presented in the next sections.

IV. HELICAL DECOMPOSITION OF SPECTRA

Following the approach presented in Ref. 1, the Fourier modes of both the velocity and the vorticity are expanded using a basis of polarized helical waves $\mathbf{h}^\pm$ defined by $i\mathbf{k}\times\mathbf{h}^\pm=\pm k\mathbf{h}^\pm$ (Refs. 7–10)

$$\mathbf{u}(\mathbf{k}) = u^+(\mathbf{k})\mathbf{h}^+ + u^-(\mathbf{k})\mathbf{h}^-, \tag{20}$$

$$\boldsymbol{\omega}(\mathbf{k}) = ku^+(\mathbf{k})\mathbf{h}^+ - ku^-(\mathbf{k})\mathbf{h}^-. \tag{21}$$

The energy and helicity carried on by the mode $\mathbf{u}(\mathbf{k})$, respectively, become

$$\mathbf{u}(\mathbf{k})\cdot\mathbf{u}^*(\mathbf{k})/2 = (|u^+(\mathbf{k})|^2 + |u^-(\mathbf{k})|^2)/2, \tag{22}$$

$$\mathbf{u}(\mathbf{k})\cdot\boldsymbol{\omega}^*(\mathbf{k})/2 = k(|u^+(\mathbf{k})|^2 - |u^-(\mathbf{k})|^2)/2. \tag{23}$$

Isotropy is again assumed and both the energy $E(k)$ and the helicity $H(k)$ spectra are considered to be functions of $k=|\mathbf{k}|$. Introducing the spectral densities of energy and helicity for the helical modes ($\pm$) yields

$$E(k) = E^+(k) + E^-(k), \tag{24}$$

$$H(k) = H^+(k) + H^-(k) = k[E^+(k) - E^-(k)]. \tag{25}$$

Their equations of evolution have exactly the structure (6). Moreover, all these quantities are dissipated through viscous effect and their linear dissipation time scale is again $\tau^{\rm diss}(\kappa)=1/(\nu k^2)$. Guessing their nonlinear characteristic time is, however, much more difficult. Indeed, nonlinear transfers can transform $E^{<+}(\kappa)$ not only in $E^{>+}(\kappa)$ but also in $E^{<-}(\kappa)$ and $E^{>-}(\kappa)$. Moreover, E^+ and E^- are not separately conserved by the nonlinear terms. Hence, invoking the equality of characteristic time scales to estimate the dissipation scales of these quantities is not necessarily justified.

It is, however, quite easy to estimate their spectra from Eqs. (13) and (18)

$$E^+(k) = \frac{C_E}{2}\varepsilon^{2/3}k^{-5/3} + \frac{C_H}{2}(\delta/\varepsilon^{1/3})k^{-8/3}, \tag{26a}$$

$$E^-(k) = \frac{C_E}{2}\varepsilon^{2/3}k^{-5/3} - \frac{C_H}{2}(\delta/\varepsilon^{1/3})k^{-8/3}, \tag{26b}$$

which are Eqs. (9) and (10) of Ref. 1. As a consequence, the leading order in k must be given by

$$E^\pm(k) = \frac{C_E}{2}\varepsilon^{2/3}k^{-5/3}, \quad H^\pm(k) = \pm\frac{C_E}{2}\varepsilon^{2/3}k^{-2/3}. \tag{27}$$

By construction, the range of validity of Eqs. (26) and (27) is the same as that of the scaling laws (13) and (18) of $E(k)$ and $H(k)$. It is therefore bounded by $k_E^D=k_H^D$.

On the other hand, formula (12) yields an estimate of the scale from which on the dissipative term must be considered in the evolution of $E^\pm$ and $H^\pm$. It leads to

$$k_{E^\pm}^\nu \propto \left(\frac{\epsilon}{\nu^3}\right)^{1/4} \tag{28}$$

and

$$k_{H^\pm}^\nu \propto \left(\frac{\delta}{\nu\epsilon^{2/3}}\right)^{3/7} \propto \left(\frac{\delta^3}{\nu^3\epsilon^2}\right)^{1/7}. \tag{29}$$

As noted by Ditlevsen and Giuliani,[1] $k_{H^\pm}^\nu < k_{E^\pm}^\nu = k_E^D$. Indeed, the helicity injection rate is at most $k_F\epsilon$, so that

$$k_{H^\pm}^\nu < \left(\frac{k_F^3\epsilon}{\nu^3}\right)^{1/7} = \left(\frac{k_F}{k_E^D}\right)^{3/7} k_E^D \tag{30}$$

And, since $k_F < k_E^D$ in the turbulent regime, $k_{H^\pm}^\nu < k_E^D$. However, there is no reason to identify the helicity dissipation scale as $k_{H^\pm}^\nu$. Clearly, the spectrum of $H^+(k)$ and $H^-(k)$ cannot deviate from the scaling $k^{-2/3}$ and fall off in the range $k_{H^\pm}^\nu < k < k_E^D$. Indeed, if these quantities decay faster than $k^{-2/3}$ after $k_{H^\pm}^\nu$, then the quantities $E^\pm(k)=\pm H^\pm(k)/k$ will decay faster than $k^{-5/3}$ and $k_{H^\pm}^\nu$ would be identified as the end of the inertial range, which is known to actually extend down to k_E^D. Considering the Eq. (10) applied to $Q=H^\pm$ allows to better understand the meaning of $k_{H^\pm}^\nu$

$$\Pi_H^{\pm<} = \delta^\pm \mp \tfrac{3}{7}C_E\nu\varepsilon^{2/3}k^{7/3}. \tag{31}$$

For low values of $k \ll k_{H^\pm}^\nu$, the fluxes are constant and equal to $\delta^\pm$. However, for $k \gg k_{H^\pm}^\nu$, the dissipation of $H^\pm$ is stronger than the injection rate $\delta^\pm$ and the nonlinear flux has to scale like $k^{7/3}$. Ditlevsen and Giuliani referred to this scale the dissipation scale of $Q^\pm$. However, as argued above, this does not correspond to the end of the helicity spectrum.

V. HELICAL SHELL MODEL ANALYSIS

Shell models are built to describe the exchange of physically relevant quantities between the various scales of a turbulent flow. The Fourier space is divided into a set of shells which are logarithmically spaced. A field (such as velocity, for instance) is represented by very few (1 or 2) complex variables in each shell. These models allow to investigate turbulence properties at a much lower numerical cost than direct numerical simulations (DNS). It must be acknowledged that DNS of the Navier–Stokes equations would be much more satisfactory. For instance, the database produced by Chen *et al.*[6] provides very interesting insights on the energy and helicity cascades. As recognized by these authors, however, the resolutions achievable in DNS did not allow for a complete analysis of these cascades. Despite the evolution in the accessible computational power, the situation has not changed drastically and, in particular, DNS are unfortunately still too limited to distinguish clearly $k^{\nu}_{H^{\pm}}$ from k^{D}_{E}. As will be shown below, Reynolds numbers, defined at scale k_F as Re $=\varepsilon^{1/3}k_F^{-4/3}\nu^{-1}$, as large as 10^7 can be reached with a shell model.

The definition of helicity, which is a pseudoscalar quantity, is not trivial in shell models as they only deal with scalar variables. However, helical shell models can be defined by using two scalar variables per shell that correspond to the amplitudes of the two eigenvectors of the curl operator. For instance, helical shell models based on this helical decomposition of Fourier modes[10] have been developed in Ref. 11. As discussed in detail in Ref. 12, such helical models can be retrieved from the helical triadic systems of the Navier–Stokes equations in helical basis. Four simple models can be expressed in a single formula

$$d_t u_n^{\pm} = W_n^{\pm} - \nu k_n^2 u_n^{\pm} + f_n^{\pm}, \tag{32a}$$

with

$$W_n^{\pm} = ik_n[(s_1\lambda - s_2\lambda^2)u_{n+1}^{\pm s_1}u_{n+2}^{\pm s_2} + (s_2\lambda - \lambda^{-1})u_{n-1}^{\pm s_1}u_{n+1}^{\pm s_2 s_1} + (\lambda^{-2} - s_1\lambda^{-1})u_{n-2}^{\pm s_2}u_{n-1}^{\pm s_1 s_2}]^*, \tag{32b}$$

where each model is obtained for one particular choice of (s_1,s_2) with $s_1,s_2=\pm 1$. In Eq. (32), the parameter λ is the logarithmic shell spacing and the wave number is defined as $k_n=k_0\lambda^n$.

In the absence of forcing and viscosity ν, the shell model (32) conserves total energy E and helicity H (Ref. 12)

$$E = \sum_{n=1}^{N} E_n, \quad H = \sum_{n=1}^{N} H_n, \tag{33}$$

where N is the number of shells in the model. The energy E_n and helicity H_n in shell n are defined as

$$E_n = E_n^+ + E_n^-, \quad E_n^{\pm} = \tfrac{1}{2}|u_n^{\pm}|^2, \tag{34}$$

$$H_n = H_n^+ + H_n^-, \quad H_n^{\pm} = \pm \tfrac{1}{2}k_n|u_n^{\pm}|^2. \tag{35}$$

Within the model, the fluxes of energy and helicity are defined as

$$\Pi_E^{<}(n) = \Pi_E^{+<}(n) + \Pi_E^{-<}(n), \tag{36}$$

$$\Pi_H^{<}(n) = \Pi_H^{+<}(n) + \Pi_H^{-<}(n), \tag{37}$$

with the following explicit expressions:

$$\Pi_E^{\pm<}(n) = -\left.\left(d_t\sum_{m=1}^{n}\frac{1}{2}|u_m^{\pm}|^2\right)\right|_{\mathrm{NL}} = -\sum_{m=1}^{n} W_m^{\pm}u_m^{\pm *} + cc, \tag{38}$$

$$\Pi_H^{\pm<}(n) = -\left.\left(d_t\sum_{m=1}^{n}\frac{1}{2}(\pm k_m)|u_m^{\pm}|^2\right)\right|_{\mathrm{NL}} = \mp\sum_{m=1}^{n} k_m W_m^{\pm}u_m^{\pm *} + cc, \tag{39}$$

where $\Pi_Q^{<}(k)$ is the flux (due to the nonlinear term) of the quantity Q leaving the region of wave numbers lower than k and $\Pi_Q^{\pm<}(k)$ is the flux leaving either the "+" or the "−" variables of wave numbers lower than k.

The GOY model used in Ref. 2 corresponds to $(s_1,s_2)=(-1,+1)$ in the helical picture (32). In this case, two uncoupled sets of variables appear, namely, $(u_1^+,u_2^-,u_3^+,\ldots)$ and $(u_1^-,u_2^+,u_3^-,\ldots)$. In the original version of the GOY model, only one of these sets is considered. Hence, in each shell n, the helicity is evaluated alternatively by H_n^+ or H_n^-, depending whether n is odd or even. The cancellation of the leading terms in Eq. (25) with the scaling (26a) and (26b) does not occur. Therefore, $H(k)$ cannot be straightforwardly obtained with a GOY model. The fluxes presented in Ref. 2 are hence closer to $\Pi_H^{\pm<}(\kappa)$ than to $\Pi_H^{<}(\kappa)$ although, *stricto sensu*, they are neither of them.

On the other hand, the developments proposed in Ref. 6 were illustrated by the SABRA (an improvement of the GOY model[13]) version of the model corresponding to $(s_1,s_2)=(+1,-1)$, in which all variables are coupled. Both H_n^+ and H_n^- are available within each shell n and so is the total helicity H_n. In Sec. VI, the work in Ref. 6 is pursued and the energy and helicity spectra and fluxes are investigated.

VI. NUMERICAL RESULTS

The computation of the averaged helicity spectra, which is the difference of its two helical components and requires the canceling of the leading terms, demands very fine time stepping. Furthermore, very long simulations are required in order to obtain enough statistics. This is probably the reason why helicity spectra are rarely reported in DNS.[14] In shell model simulations, helicity spectra have been obtained in Ref. 15. Very long and accurate integration of the shell model (32) with $(s_1,s_2)=(+1,-1)$ have been performed. In these simulations, the forcing is concentrated on one single shell (the fourth) and provides constant energy and helicity injection rates. The rate of energy injection within the "±" variables is denoted $\varepsilon^{\pm}$ and the one of total energy $\varepsilon=\varepsilon^+ + \varepsilon^-$. The rate of helicity injection is therefore $\delta=\delta^+ + \delta^-$ with $\delta^{\pm}=\pm k_F\varepsilon^{\pm}$.

In Figs. 1 and 2, the results are presented for, respectively, a helical and a nonhelical case. The parameters are

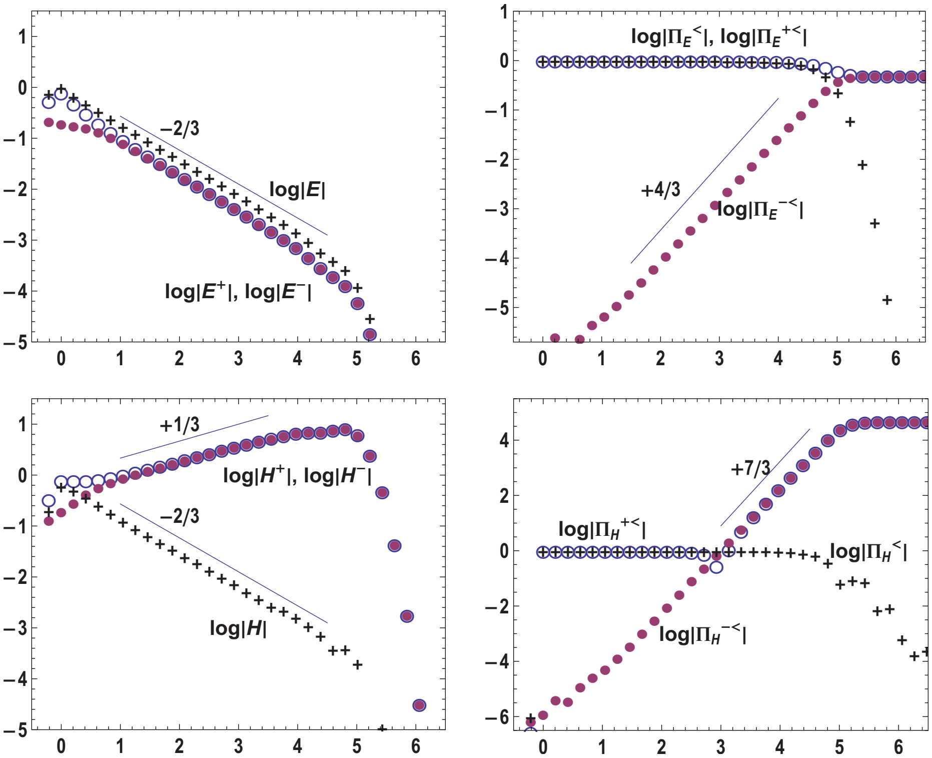

FIG. 1. (Color online) Helical case: $\varepsilon=\varepsilon^{+}=1$, $\delta^{-}=\varepsilon^{-}=0$. Energy and helicity plots are, respectively, represented on the left and right columns vs log k. The positive and negative helical modes are denoted by ○ and ● and the sum of both modes by +. The spectra (fluxes) are represented in the top (bottom) row.

$\nu=10^{-7}$ and $\lambda=(1+\sqrt{5})/2$. The shell are labeled from -2 to 37 with $k_n=\lambda^n$. The total number of shells is thus $N=40$ and the forcing is concentrated in the third shell so that $k_F=1$. For the helical case, $\varepsilon^{+}=\varepsilon=1$, implying $\delta=\delta^{+}=1$ and $\varepsilon^{-}=-\delta^{-}/k_F=0$. For the nonhelical case, $\varepsilon^{+}=\varepsilon^{-}=1/2$, implying $\varepsilon=1$, $\delta^{+}=-\delta^{-}=1/2$, and $\delta=0$. In each figure, the left and right columns correspond, respectively, to energies and helicities.

The spectra are plotted in log-log frames (upper row). Energies $E(k_n)$ and $E^{\pm}(k_n)$ scale in $k_n^{-2/3}$ corresponding to power spectral densities in $k^{-5/3}$ in agreement with Eqs. (13) and (27). Helicities $H^{\pm}(k_n)$ scale in $k_n^{1/3}$ corresponding to power spectral densities in $k^{-2/3}$ in agreement with Eq. (27). In the helical case, the total helicity $H(k_n)$ scales in $k_n^{-2/3}$ corresponding to a power spectral density in $k^{-5/3}$ in agreement with Eq. (18). In the nonhelical case, $H(k_n)$ is the sum

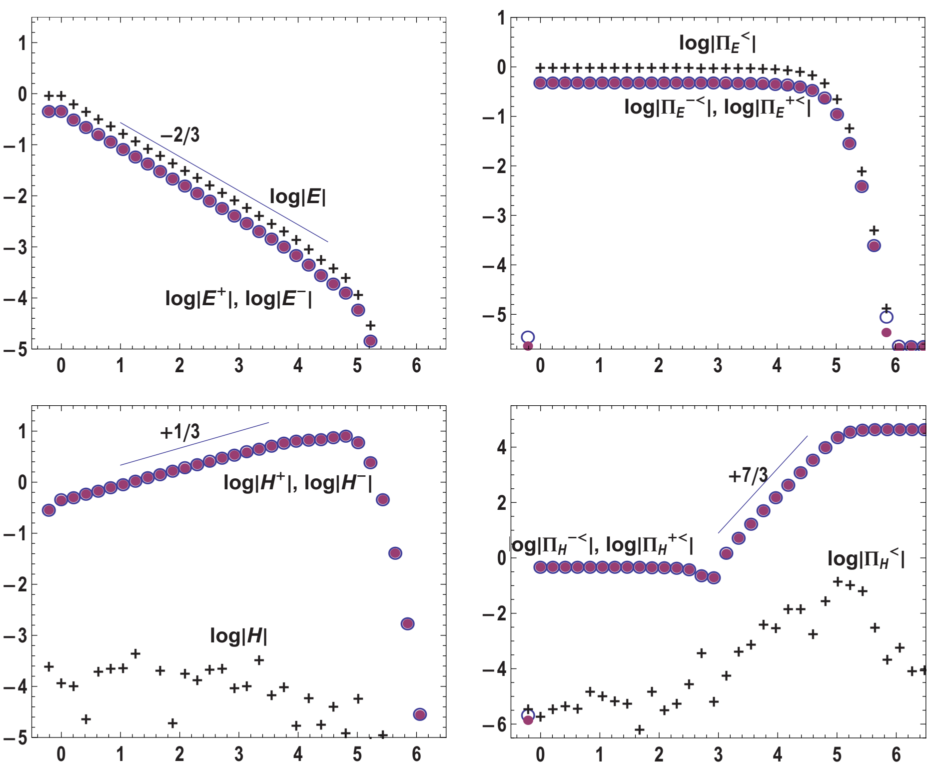

FIG. 2. (Color online) Same as Fig. 1 for the nonhelical case $\varepsilon^{-}=\varepsilon^{+}=1/2$, $\delta=0$.

of two opposite quantities $H^{\pm}(k)$ and has no clear scaling. Compared to $H^{\pm}(k_n)$, it can be considered as negligible, in agreement with Eq. (18), taking $\delta=0$. Note that all spectra manifestly extend up to the Kolmogorov scale $k_E^D \sim 10^5$.

The nonlinear fluxes are plotted in log-log frames (lower row). For the helical case, the total energy flux as well as the energy flux of E^+ are constant and dominated by $\epsilon=1$ up to the Kolmogorov scale. On the contrary, the flux of E^- has no component corresponding to the injection since $\epsilon^-=0$ so that its spectrum is dominated for low k by the viscous term and is proportional to $k^{4/3}$. The viscous scale $k_{H^\pm}^\nu$ is clearly identified on the helicity flux for H^+. For $k < k_{H^\pm}^\nu$, the flux is constant and dominated by δ^+, while for $k > k_{H^\pm}^\nu$ the injection is subleading and the flux scales like $k^{7/3}$. Remarkably, the viscous scale $k_{H^\pm}^\nu$ is also very clearly observed even in the nonhelical case.

VII. CONCLUSION

The present study has allowed to identify two different length scales related to the dissipation of a quadratic quantity Q in Navier–Stokes turbulence. The first one is the traditional dissipation scale that marks the end of the power law in the spectrum of Q due to the dominant effect of the viscosity. The second scale, referred to as the viscous scale, corresponds to the beginning of the range in which viscous effect have to be taken into account. Clearly, for the kinetic energy, the viscous and the dissipation scales coincide. However, for nonconserved quantities, such as the positive and negative part of the helicity, these two scales are different. Although the viscous scale cannot be measured from the spectra, it is easily identified from the nonlinear fluxes. This has been shown using shell models.

This approach reconcile the analysis of Refs. 1, 2, and 6. Strictly speaking, the scale $k_{H^\pm}^\nu$ cannot be interpreted as the dissipation scale for helicity. Both direct shell model integration and helical components analysis show that the helicity cascade develops down to the Kolmogorov scale. However, this scale is indeed relevant in the analysis of the nonlinear flux of helicity and plays a role even when the flow is globally nonhelical.

Beyond the issue of helicity dissipation scale which is now clarified, this study stresses how much caution is required when studying the effect of helicity on turbulence dynamics with a GOY model.[16] Other models such as the one used here or those presented in Ref. 12 or Ref. 15 are highly preferable.

As suggested by Brissaud *et al.*,[17] the assumption that both energy and helicity have the same spectrum (13–18) might not hold if these quantities are forced at different scales. Although such a problem would be out of the scope of the present paper, helical shell models are perfectly adapted to study these situations. For instance, a rate of energy injection could be prescribed at a given scale $k_{n_f}^{-1}$ without injecting helicity, while an injection of helicity (without injection of energy) could be enforced at another scale $k_{n_f'}^{-1}$ by transferring energy from $u_{n_f'}^-$ to $u_{n_f'}^+$. In particular, between $k_{n_f}^{-1}$ and $k_{n_f'}^{-1}$ the energy and helicity spectra should be different.

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