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Effect of Absorption on Energy Partition of Elastic Waves in the Seismic Coda

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Abstract The stabilization of the ratio of *P* to *S* energy in the multiple scattering regime was first predicted theoretically by Weaver (1982) and has been recently observed in the high-frequency coda of earthquakes by Shapiro *et al.* (2000). The goal of this note is to extend previous theoretical results to the case where anelasticity is present in the medium. Our study demonstrates that even if there is preferential absorption of one of the modes (*P* or *S*), a stabilization occurs in the multiple scattering regime. The new equilibrium ratio is shifted in favor of the mode that is least absorbed but rarely differs by more than 15% from the ratio predicted for purely elastic media.

Introduction

Recently, Shapiro *et al.* (2000) reported observations of stabilization of the shear (*S*) to compressional (*P*) energy of the wave field in the coda of high-frequency local earthquakes in Mexico. This observation can be explained by the equal distribution of the energy among all the normal modes of the system, termed the equipartition principle. Equipartition is expected to occur when multiple scattering of waves is dominant and is a marker of the diffusive propagation regime. In the case of an infinite elastic body, the normal modes are just plane waves, and the equipartition principle predicts (Weaver, 1982):

$$\frac{E^S}{E^P} = \frac{2\alpha^3}{\beta^3}, \quad (1)$$

where E^S and E^P denote the energy density of *P* and *S* waves, respectively. This equation can be obtained by counting the number of compressional and shear modes in an infinite solid. The factor of 2 accounts for the existence of 2 degrees of freedom for shear waves. A rigorous derivation of this equation has been provided by Ryzhik *et al.* (1996). For crustal materials, we expect a ratio of about 10.4 when equipartition is reached. This is rather different from the value of 7 observed by Shapiro *et al.* (2000) in Mexico. One origin of this discrepancy may be the dissipation of elastic waves. In this note, we explore the effect of absorption on equipartition of elastic waves. We show in particular that the ratio measured by Shapiro *et al.* (2000) cannot be satisfactorily explained by anelasticity.

Theory

As explained above, equation (1) can be derived by counting normal modes. This approach cannot be generalized to the case where anelasticity is present in the system. Moreover, it is *a priori* not even clear whether an equipartition sets in at all, and if so, whether the equation above is still valid. To answer these questions, we will use radiative transfer theory. The elastic radiative transfer equation has been derived independently by Weaver (1990) and Ryzhik *et al.* (1996). We refer to Papanicolaou *et al.* (1996), Turner (1998), and Sato and Fehler (1998) for a detailed seismological account of the theory. The elastic radiative transfer equation reads:

$$\mathbf{c}^{-1}\frac{\partial \mathbf{I}(\mathbf{r}, \mathbf{s}, t)}{\partial t} + \mathbf{s}\cdot\nabla_r \mathbf{I}(\mathbf{r}, \mathbf{s}, t) = -\mathbf{l}^{-1} - \mathbf{l}_a^{-1} + \frac{\mathbf{l}^{-1}}{4\pi}\int d^2\mathbf{s}'\mathbf{I}(\mathbf{r}, \mathbf{s}, t)\mathbf{P}(\mathbf{s}, \mathbf{s}). \quad (2)$$

$\mathbf{I}(\mathbf{r}, \mathbf{s}', t)$ is the elastic Stokes vector at point $\mathbf{r}$ in space and time t; $\mathbf{P}(\mathbf{s}, \mathbf{s}')$ is the Mueller matrix that governs the scattering of intensity from space direction $\mathbf{s}'$ to space direction $\mathbf{s}$; $\mathbf{c}$ is the diagonal velocity matrix, $\mathbf{l}_a$ is the diagonal absorption matrix; $\mathbf{l}$ is the diagonal mean free path matrix. The Stokes vector $\mathbf{I} = (I_P, I_{S_x}, I_{S_y}, U, V)$ contains all information about the energy and state of polarization of the scattered wavefield. I_P, $(I_{S_x} + I_{S_y})$ describe the total intensity of *P* and *S* waves, respectively, while the two additional parameters *U* and *V* enable to represent the polarization of the *S* wave. The different Stokes parameters are coupled through scattering, which makes the Mueller matrix $\mathbf{P}(\mathbf{s}, \mathbf{s}')$ a nondiagonal matrix. The last term on the right-hand side represents the scattered Stokes vector. Equation (2) is far too detailed for our purposes. Following Turner and Weaver (1994), we integrate the radiative transfer equation over the whole solid angle and over all space to obtain a balance equation for the global distribution of energy among P and S waves. Upon integration, the first three Stokes parameters, I_P, I_{S_x}, I_{S_y}, decouple from the last two parameters, U, V, describing the polarization. This property stems from the adopted statistical isotropy of the random medium, which implies that the angularly integrated Mueller matrix $\mathbf{M} = \int_{4\pi} \mathbf{P}(\mathbf{s}, \mathbf{s}')\mathbf{d}^2\mathbf{s}'$ has the following properties:

$$M_{14} = M_{15} = M_{24} = M_{25} = M_{34} = M_{35} = M_{31} = M_{32} = M_{41} = M_{51} = M_{42} = M_{52} = 0 \quad (3)$$

Note that in a statistically isotropic medium, the M_{ij} are independent of the outgoing scattering direction $\mathbf{s}$. The 3×3 system of differential equations for the intensities of P and S waves can be further reduced by summing up the contributions of both shear polarizations. This last step does not involve any approximation because in a statistically isotropic medium the following symmetry relations hold (Turner and Weaver 1994):

$$M_{12} = M_{13}, M_{21} = M_{31}, M_{32} = M_{23}. \quad (4)$$

In order to write down the global balance equation and discuss its implications, we need to introduce a few notations: τ^{PS} (τ^{SP}), the P (S) to S (P) scattering mean free time; and τ_a^P (τ_a^S), the absorption time for P (S) waves.

We are now ready to introduce the evolution equation of the total energy density $\mathbf{E} = (E^P, E^S)$:

$$\frac{d\mathbf{E}}{dt} = \Delta\mathbf{E}. \quad (5)$$

The 2×2 matrix Δ can be written as:

$$\begin{pmatrix} -\left(\frac{1}{\tau^{PS}} + \frac{1}{\tau_a^P}\right) & \frac{1}{\tau^{SP}} \\ \frac{1}{\tau^{PS}} & -\left(\frac{1}{\tau^{SP}} + \frac{1}{\tau_a^S}\right) \end{pmatrix}, \quad (6)$$

expressing the exchange of energy between P and S waves in the random medium, and the effect of absorption. The solution of the system (6) will lead us to the equipartition relation and the characteristic time to reach equipartition.

Results

Consider the case where there is no absorption ($1/\tau_a^P = 1/\tau_a^S = 0$). We find that the eigenvalues λ_0, λ_1 and corresponding eigenvectors V_0, V_1 of equation (6) are:

$$\lambda_0 = 0, \mathbf{V}_0 = \left(1, 2\frac{\alpha^3}{\beta^3}\right) \quad (7)$$

$$\lambda_1 = -\left(\frac{1}{\tau^{PS}} + \frac{1}{\tau^{SP}}\right), \mathbf{V}_1 = (1, -1). \quad (8)$$

The first eigenvalue and eigenvector clearly correspond to the equipartition relation (1). The second eigenvalue gives us a characteristic equipartition time between P and S waves. These results were also obtained by Turner and Weaver (1994).

Consider the case where absorption is present. In the limit where scattering is very weak compared to absorption (τ_a^P, $\tau_a^S \ll \tau^{SP}$, τ^{PS}), the matrix Δ becomes approximately diagonal, leading us to the new eigenvalues and eigenvectors:

$$\lambda_0' = -\frac{1}{\tau_a^P}, \mathbf{V}_0' = (1, 0) \quad (9)$$

$$\lambda_1' = -\frac{1}{\tau_a^S}, \mathbf{V}_1' = (0, 1), \quad (10)$$

which shows that the energy of each mode is dissipated before an equipartition can set in. The observation of Shapiro *et al.* (2000) therefore imply that the anelasticity of the crust must be relatively weak. We assume henceforth that the dissipation times are larger than the scattering mean free times. Mathematically, this allows us to give relatively simple expressions for the eigenvalues and eigenvectors of a dissipative medium, correct to first order in the small parameters τ^{PS}/τ_a^P, τ^{SP}/τ_a^S:

$$\lambda_0'' = -\frac{\frac{1}{\tau^{PS}\tau_a^S} + \frac{1}{\tau^{SP}\tau_a^P}}{\frac{1}{\tau^{PS}} + \frac{1}{\tau^{SP}}}$$

$$\mathbf{V}_0'' = \left(1, 2\frac{\alpha^3}{\beta^3}\left(1 + \frac{\frac{1}{\tau_a^P} - \frac{1}{\tau_a^S}}{\frac{1}{\tau^{PS}} + \frac{1}{\tau^{SP}}}\right)\right). \quad (11)$$

The second eigenvalue λ_1'' is larger in absolute value and corresponds to an eigenfunction that decays on a much faster timescale:

$$\lambda_1'' = -\left(\frac{1}{\tau^{PS}} + \frac{1}{\tau^{SP}}\right)\left(1 + \frac{\frac{1}{\tau_a^P\tau^{PS}} + \frac{1}{\tau_a^S\tau^{SP}}}{\left(\frac{1}{\tau^{PS}} + \frac{1}{\tau^{SP}}\right)^2}\right)$$

$$\mathbf{V}_1'' = \left(1, -1 + \frac{\frac{1}{\tau_a^P} - \frac{1}{\tau_a^S}}{\frac{1}{\tau^{PS}} + \frac{1}{\tau^{SP}}}\right). \quad (12)$$

The general solution of the linear system (6) can be expressed as:

$$\mathbf{E} = C_0 \mathbf{V}_0'' e^{\lambda_0'' t} + C_1 \mathbf{V}_1'' e^{\lambda_1'' t}, \tag{13}$$

where C_0 and C_1 are constants determined by the initial conditions. Since λ_1'' is larger in absolute value than λ_0'', we expect for sufficiently large lapse times—of the order of $-1/\lambda_1''$—that only the first term in equation (13) contributes. The physical interpretation of equations (11) and (13) is thus clear. The new equipartition relation is just the product of the nondissipative E_S/E_P ratio, times a correction term. If the absorption time of P waves is equal to the absorption time of S waves, the equipartition relation (1) is unaffected, since both modes are equally absorbed. However, if one of the modes is more absorbed than the other, say $\tau_a^P > \tau_a^S$, then the ratio E^S/E^P reaches a new equipartition constant, which is shifted in favor of P waves. The opposite conclusion applies when $\tau_a^S > \tau_a^P$.

To illustrate these results, let us consider the case of an explosive source. At $t = 0$, the initial conditions read: $E^P = 1$, $E^S = 0$. We assume a Poisson solid and choose some reasonable values of intrinsic attenuation and scattering mean free path: $\tau^{SP} = 50$ sec, $Q_i^P = 2\pi\tau_a^P = 1000$, $Q_i^S = 2\pi\tau_a^S = 300$, and a central frequency of 1 Hz is implied. Since observations by Shapiro *et al.* (2000) report equipartition values close to 7, a stronger attenuation is imposed to S waves. The results are presented in Figure 1, where we show the time evolution of E^P, E^S and E^P/E^S. The characteristic time of equipartition τ_{eq} can be estimated from the eigenvalue λ_1''. For the above mentioned value of τ^{SP}, Q^S and Q^P, we find $\tau_{eq} \cong 10$ sec. We emphasize that this value does not give us any indication of the local equipartition time. The initial ratio of P and S energies radiated by a point shear dislocation is given by $E^P/E^S = \frac{2}{3}(\beta/\alpha)^5$ and roughly equals 0.05 for a Poisson solid (Sato and Fehler, 1998). The time evolution of the P and S energies is shown in Figure 2. The comparison with Figure 1 shows that a rapid equipartition is favored by shear sources. For the set of parameters assumed above, the predicted equipartition ratio is about 9.25, independent of the type of source.

To investigate the impact of absorption on the equipartition value, we introduce δ_{eq} as the relative change of the ratio E^S/E^P at equipartition in a dissipative medium, compared to its value in a purely elastic medium:

$$\delta_{eq} = \frac{1}{2\alpha^3/\beta^3 + 1} \times \left(1 - \frac{\tau_a^P}{\tau_a^S}\right) \times \frac{\tau^{SP}}{\tau_a^P}. \tag{14}$$

In deriving this last equation, we have used the reciprocity relation $\tau^{SP}/\tau^{PS} = 2\alpha^3/\beta^3$. The first factor on the left-hand side is typically 0.1. The last factor characterizes the importance of absorption and should not be too large, otherwise, as explained above, the elastic energy would be dissipated before equipartition sets in. The τ^{SP}/τ_a^P ratio is therefore assumed to range from 0 to 0.4, which we believe, corresponds to reasonable values of anelasticity. The sign of δ_{eq} depends on the second factor, which measures the relative absorption of S waves as compared to P waves. Since the observations of Shapiro *et al.* (2000) show that the equipartition ratio is shifted in favor of P waves, we impose the condition $\tau_a^P \geq \tau_a^S$, and let the second factor of equation (14) range from 0 to -9, which encompasses most situations likely to be met in practice. The lower bound corresponds to a medium where the dissipation of shear energy is roughly one order of magnitude stronger than the dissipation of compressional energy. A contour plot of δ_{eq} as a function of $(1 - \tau_a^P/\tau_a^S)$ and τ^{SP}/τ_a^P is shown in Figure 3. Typically, the correction term (14) rarely exceeds 15%, which is twice as less as the value required to explain the observations. Only in a rather anelastic solid where, in addition, S waves would be dissipated

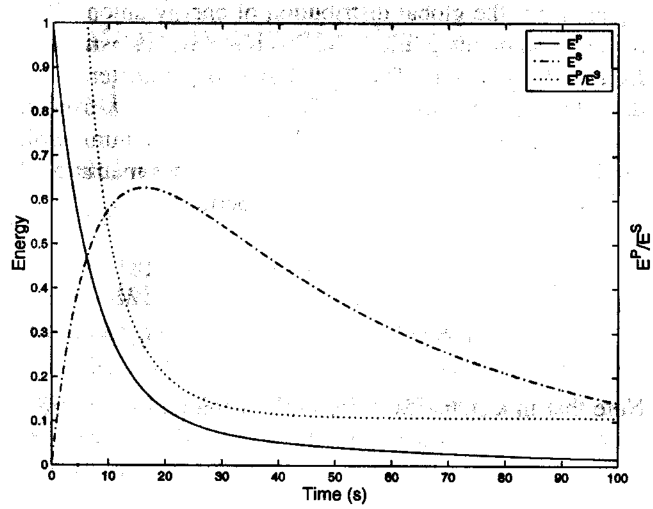

Figure 1. Dependence of E^P, E^S, and E^P/E^S on time for an explosive source.

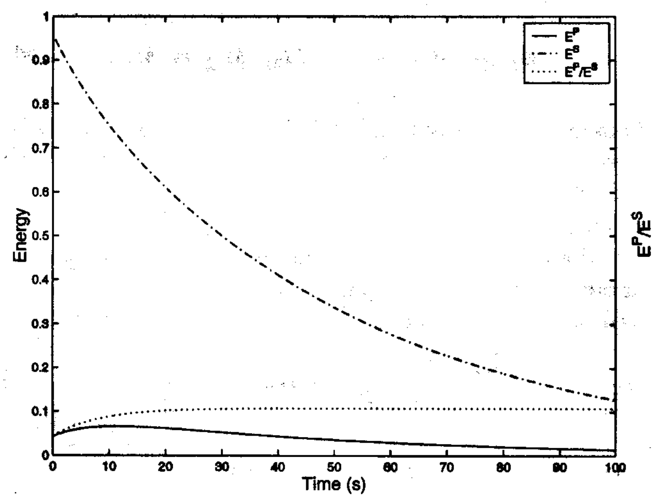

Figure 2. Dependence of E^P, E^S, and E^P/E^S on time for a shear dislocation.

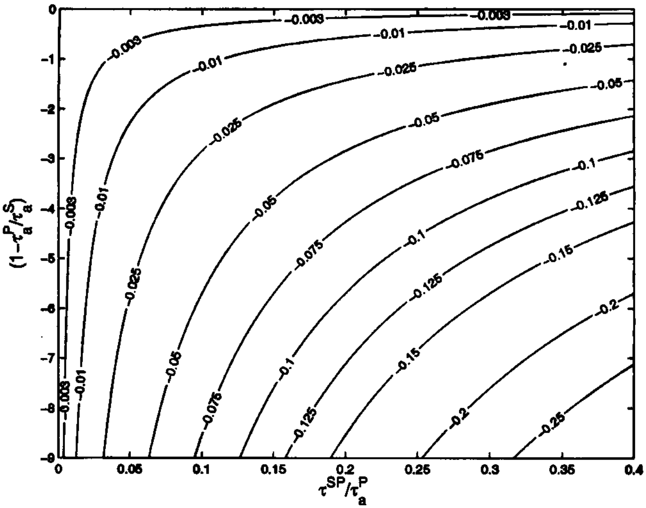

Figure 3. Contour plot of the relative change of the equipartition ratio E^S/E^P in dissipative media compared to purely elastic media, as a function of $(1 - \tau_\alpha^P/\tau_\alpha^S$ and τ^{SP}/τ_α^P. The perturbation of the E^S/E^P ratio rarely exceeds 15%.

much more strongly than *P* waves, could one find an equipartition ratio in reasonable agreement with the value reported by Shapiro *et al.* (2000). It seems therefore necessary to look for alternative physical mechanisms. The interaction of incident and reflected waves as well as the excitation of surface waves may affect severely the local equipartition ratio measured at the surface. This important problem goes far beyond the goal of this note and will be addressed in a separate publication. The present analysis may be relevant to other cases where elastic wave diffusion is hampered by dissipation.

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