

# An optimal scale separation for a dynamo experiment

Franck Plunian<sup>a)</sup>

Laboratoires des Écoulements Géophysiques et Industriels, Boîte Postale 53,  
38041 Grenoble Cedex 9, France

(Received 30 September 2004; accepted 14 January 2005; published online 5 April 2005)

Scale separation between the flow and the magnetic field is a common feature of natural dynamos. It has also been used in the Karlsruhe sodium experiment in which the scale of the magnetic field is roughly seven times larger than the scale of the flow [R. Stieglitz and U. Müller, "Experimental demonstration of the homogeneous two-scale dynamo," *Phys. Fluids* **13**, 561 (2001)]. Recently, Fauve and Pétrélis [*Peyresq Lectures on Nonlinear Phenomena*, edited by J. Sepulchre (World Scientific, Singapore, 2003), p. 1] have shown that the power needed to reach the dynamo threshold in a dynamo experiment increases with the scale separation in the limit of large scale separation. With a more elaborate method based on subharmonic solutions [F. Plunian and K.-H. Rädler, "Subharmonic dynamo action in the Roberts flow," *Geophys. Astrophys. Fluid Dyn.* **96**, 115 (2002)], we show, for the Roberts flow, the existence of an optimal scale separation for which this power is minimum. Previous results obtained by Tilgner ["A kinematic dynamo with a small scale velocity field," *Phys. Lett. A* **226**, 75 (1997)] with a completely different numerical method are also reconsidered here. Again, we find an optimal scale separation in terms of minimum power for dynamo action. In addition we find that this scale separation compares very well with the one derived from the subharmonic solutions method. © 2005 American Institute of Physics. [DOI: 10.1063/1.1884065]

We consider a dynamo experiment with a horizontal scale separation between the characteristic scale  $l$  of the flow and the size  $L$  of the container as, for example, in the Karlsruhe experiment.<sup>1</sup> In addition, the flow is assumed to have a geometry which can lead to the self-excitation of a magnetic field at the size of the container. For that we consider a Roberts<sup>2</sup> flow within a cubic box as in Ref. 3. In the  $x$  and  $y$  directions (where  $x$ ,  $y$ , and  $z$  are the Cartesian coordinates), the size of a flow cell is  $l \times l$  and the size of the box is  $L \times L$ . In the perpendicular direction  $z$  the flow cells and the box have the common size  $H$ . Then the number of cells is  $N_c = L^2/l^2$ .

Following Ref. 4, we assume that the power  $P$  is dissipated by turbulence, leading to  $P = \rho L^2 H U^3 / l$  where  $\rho$  is the density and  $U$  the characteristic speed of the fluid. Defining the magnetic Reynolds number by  $R_m = Ul/\eta$ , where  $\eta$  is the magnetic diffusivity of the fluid, we find after some simple algebra that

$$P = \rho \eta^3 \frac{H}{L^2} N_c^2 R_m^3. \quad (1)$$

As a preliminary step, we assume that the first-order smoothing approximation is valid (a sufficient condition being  $R_m \leq 1$ ). Then we have the relation  $\eta b/l^2 \approx UB/l$  between the small scale  $b$  and the large scale  $B$  magnetic field intensities. Furthermore at the onset we have the following relation  $\alpha K = \eta K^2$  derived from the mean part of the induction equation and where  $\alpha$  corresponds to the anisotropic  $\alpha$  effect produced by the Roberts flow,  $K$  being the vertical

wave number of the magnetic field. Here we take  $K = 1/H$ , leading to  $\alpha = \eta/H$ . Writing that the mean electromotive force  $Ub$  is equal to  $\alpha B$  leads to the following relation:

$$U\sqrt{Hl}/\eta \approx 1. \quad (2)$$

Then we can show that  $\sqrt{(H/L)R_m} \approx N_c^{-1/4}$ , leading to

$$P \approx \frac{\rho \eta^3}{\sqrt{LH}} N_c^{5/4}. \quad (3)$$

From this simple estimate we conclude that the power consumption increases with the number of cells which is not in favor of scale separation. This was found previously by Fauve and Pétrélis<sup>4</sup> for a scale separation in the three Cartesian directions (instead of only two in our case), leading to a different scaling  $N_c^{5/6}$ . Both estimates are based on the first-order smoothing approximation which has been proved to be too simplistic in the theoretical predictions of the Karlsruhe experiment. Therefore we reconsider this problem below in the light of the subharmonic solutions as studied in Ref. 3.

The original Roberts<sup>2</sup> flow is defined by

$$\mathbf{U} = U \left( \sin \frac{y}{L_U}, \sin \frac{x}{L_U}, \cos \frac{x}{L_U} - \cos \frac{y}{L_U} \right) \quad (4)$$

and the relations between the dimensions defined above and those defined in Ref. 3 are  $l = \pi\sqrt{2}L_U$ ,  $L = \pi L_B$ ,  $N = L_B/L_U$ ,  $N_c = N^2/2$ , and  $R_m^* = R_m/(\pi\sqrt{2})$  where  $R_m^* = UL_U/\eta$  is the magnetic Reynolds number defined in Ref. 3. For a given value of  $N$ , we look for the subharmonic solution embedded in the box of size  $L \times L \times H$  and corresponding to the dimensionless wave numbers  $f = 1/N$  in the horizontal directions and

<sup>a)</sup>Electronic mail: Franck.Plunian@hmg.inpg.fr; URL: <http://legi.hmg.inpg.fr/~plunian>

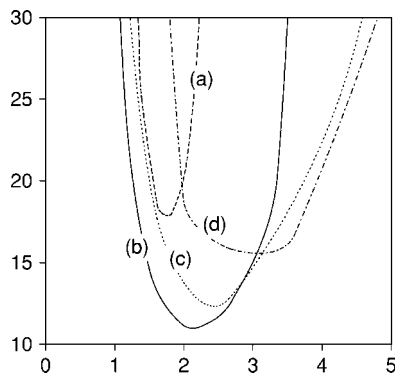


FIG. 1. The consumption power  $PH$  (in kW m) vs  $L/H$  for different numbers of cells (a)  $N_c=8$ , (b)  $N_c=18$ , (c)  $N_c=32$ , (d)  $N_c=50$ .

$k=(1/N)(L/H)$  in the vertical direction. Then (1) writes in the form

$$PH = \rho \eta^3 \frac{\pi^3}{\sqrt{2}} \left( \frac{N}{k} \right)^2 (R_m^*)^3. \quad (5)$$

For a given value of  $N$  the critical  $R_m^*$  versus  $k$  has been plotted in Fig. 4 of Ref. 3. Then replacing  $N$ ,  $k$ , and  $R_m^*$  in (5) we can calculate the corresponding power  $P$  times  $H$ . For  $\rho=10^3$  and  $\eta=0.1$  we plot, in Fig. 1,  $PH$  (in kW m) versus  $kN=L/H$  for different values of  $N_c$ . We find that the minimum value of  $PH$  is obtained for  $L/H=2.16$  and  $N_c=18$  cells. The case of the Karlsruhe experiment corresponds approximatively to  $L/H=2$  and  $N_c=50$  for which we find  $PH=19$  kW m. For the same value of  $L/H$  but for  $N_c=18$  cells, the power consumption is reduced roughly by a factor 2. In Fig. 2,  $PH$  is plotted versus  $N_c$  for  $L/H=2$  (full curve). We see that there is indeed a minimum around  $N_c=20$  and that at large  $N_c$ ,  $PH$  increases with  $N_c$  as predicted by (3). In a previous study,<sup>5</sup> Tilgner calculated the critical magnetic Rey-

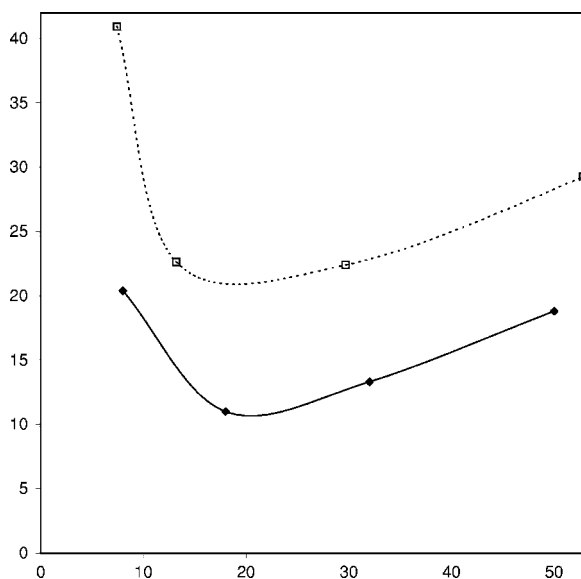


FIG. 2. The consumption power  $PH$  (in kW m) vs the cells number  $N_c$ . The full (dotted) curve is derived from Ref. 3 (from Ref. 5) for  $L/H=2$  ( $R/H=1$ ).

nolds number for the Karlsruhe experiment geometry, varying the number of cells inside the device (Fig. 4 of Ref. 5). The resolution was made with a completely different method than the one used in Ref. 3 and it is then of interest to reconsider the results of Ref. 5 in terms of power consumption and see how they compare to our results. For that we need to make preliminary correspondance between our present notations and those used in Ref. 5. In Ref. 5 the flow container is a cylinder then the consumption power, instead of (1), writes in the form  $PH = (\rho \eta^3 / \pi) (H/R)^2 N_c^2 R_m^3$  where  $R$  is the cylinder radius. In Ref. 5 we have  $l = (8/4.1)(R/N_{T97})$  where we call here  $N_{T97}$  the parameter  $N$  of Ref. 5. This leads to a number of cells  $N_c = \pi R^2 / l^2 = 0.825 N_{T97}^2$ . Furthermore the magnetic Reynolds number in Ref. 5 is defined by  $R_{mT97} = U r_0 / \eta$  where  $r_0 = \sqrt{1.25} R$  is the radius of the conducting sphere in which the cylinder is embedded. This leads to  $R_m = (1.745 / N_{T97}) R_{mT97}$ . Finally using the results from the Fig. 4 of Ref. 5, the consumption power is plotted versus the cells number on Fig. 2 (dashed curve). We find that there is again an optimal scale separation for which the dissipated power is minimum and again it corresponds to  $N_c$  close to 20. Furthermore the levels of power are of the same order of magnitude. We could not expect better agreement as the geometries and boundary conditions of Refs. 3 and 5 are really different. Now considering the design of the Karlsruhe experiment, most of the dissipation power occurs in the pieces of pipes which redirect the flow into neighboring cells at the end of each cell. The dissipation scaling in there is somewhat slower than  $U^3$  but, most importantly, it is not proportional to the volume of the experiment. Therefore the scale separation of that experiment was guided by the characteristics of the available pumps<sup>6</sup> in order to minimize the critical  $R_m$ , instead of minimizing the dissipated power, leading to  $N_c=52$  (or alternatively to  $N_{T97}=8$  corresponding to the minimum critical  $R_m$  in Ref. 5). Therefore the criterion that we derived here is relevant for propeller driven experiments, but the requirements are more complicated (and also less universal) for pump driven experiments.

The author is indebted to Jean-François Pinton for having suggested this work, to Stephan Fauve for stimulating discussions at the Isaac Newton Institute during the Workshop on Magnetohydrodynamics of Stellar Interiors, and to Andreas Tilgner for useful comments concerning the design of the Karlsruhe experiment.

<sup>1</sup>R. Stieglitz and U. Müller, "Experimental demonstration of the homogeneous two-scale dynamo," *Phys. Fluids* **13**, 561 (2001).

<sup>2</sup>G. O. Roberts, "Dynamo action of fluid motions with two-dimensional periodicity," *Philos. Trans. R. Soc. London* **271**, 411 (1972).

<sup>3</sup>F. Plunian and K.-H. Rädler, "Subharmonic dynamo action in the Roberts flow," *Geophys. Astrophys. Fluid Dyn.* **96**, 115 (2002).

<sup>4</sup>S. Fauve and F. Pétrélis, "Effect of turbulence on the onset and saturation of fluid dynamos," *Peyresq Lectures on Nonlinear Phenomena*, edited by J. Sepulchre (World Scientific, Singapore, 2003), pp. 1–64.

<sup>5</sup>A. Tilgner, "A kinematic dynamo with a small scale velocity field," *Phys. Lett. A* **226**, 75 (1997).

<sup>6</sup>F. H. Busse, U. Müller, R. Stieglitz, and A. Tilgner, "A two scale homogeneous dynamo: An extended analytical model and an experimental demonstration under development," *Magnetohydrodynamics (N.Y.)* **32**, 259 (1996).