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Chaotic behaviour of the Rikitake dynamo with symmetric mechanical friction and azimuthal currents

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We propose a new description of the two-disc dynamo, considering azimuthal currents in the rotating discs, as proposed by H. K. Moffatt for the one-disc dynamo, in addition to symmetric mechanical friction applied to the discs, whose neglect constitutes a fundamental error, as shown by R. Hide. The linear stability of the steady state is analysed and it is shown that the system presents chaotic reversals, depending on the parameters of the problem.

Keywords: dynamo effect; nonlinear instability; disc dynamo; Rikitake dynamo system; instability analysis; electrodynamical instability

1. Introduction

The one-disc (Bullard 1955) and two-disc (Rikitake 1958) dynamo systems have been widely investigated. The Bullard model, known mainly for its educational interest, presents the typical features of a fluid dynamo (Moreau 1990). The Rikitake model presents unstable solutions with chaotic reversals. Its analogy with the geodynamo has been studied, even though this simple model cannot describe the more complex MHD processes of the geodynamo, the geomagnetic data and the paleomagnetic inversion. Its interest is more historical, and also demonstrative in explaining nonlinear coupling, stability and chaos (see Cook & Roberts 1970; Cook 1972; Ito 1980; Ershov *et al*. 1989). Some new related simple idealized models for the geodynamo have been proposed recently (Chui & Moffatt 1993; Hide *et al*. 1996).

The conventional description of the one-disc dynamo has been shown to be misleading from the fundamental point of view. Indeed, Moffatt (1979) observed that if the electric current in the disc flows in a purely radial manner, then the magnetic flux through the disc has exponential growth, even within the limit of perfect disc conductivity. In this case, this violates the fundamental result of electromagnetic theory demanding that the flux through any closed curve moving with the conductor must be conserved. To remove this contradiction, Moffatt considered an azimuthal current distribution in the disc itself. For easier modelling, he segmented the disc by insulating foils, in such a way that the current is forced to flow radially, except in the neighbourhood of the disc's rim, where it is azimuthal. Moffatt studied this system neglecting the mechanical friction on the disc in comparison with ohmic losses.

Recently, Hide (1995) showed that this latter assumption (neglecting the mechanical friction) is in fact unwarranted for the two-disc dynamo. Furthermore, accord-

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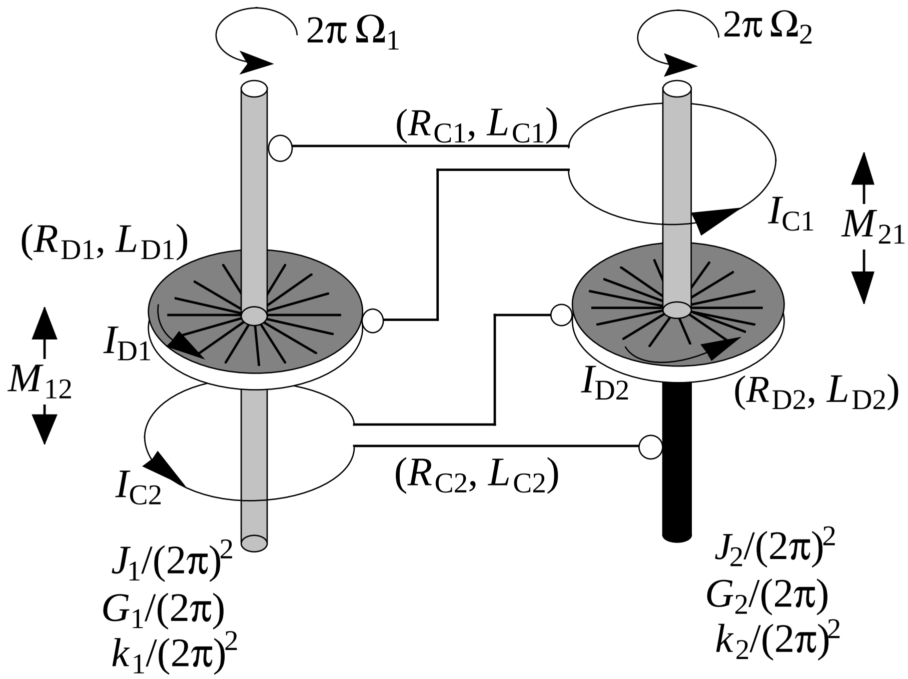

Figure 1. Description of the two-disc dynamo model.

ing to Hide, symmetric mechanical friction 'can render the Rikitake dynamo structurally unstable and consequently incapable of producing chaotic oscillations'. The main objective of this paper is to see whether this statement remains valid when azimuthal currents in the discs are considered in addition to symmetric mechanical friction. Moreover, the one-disc dynamo studied by Moffatt will also be discussed, but this time with mechanical friction, in order to see if unstable solutions are still possible.

In § 2, a mathematical model of a non-symmetric two-disc dynamo, including mechanical friction and azimuthal currents, is presented. It is followed, in § 3, by a linear stability analysis of the steady state for two identical discs. In § 4, the formulation and analysis of the one-disc dynamo case are derived.

2. Formulation of the two-disc dynamo

(*a*) *Characteristics of the system*

The system is composed of two coupled disc dynamos (figure 1). For easier notation, equations are written, whenever possible, for the disc dynamo i ($i = 1, 2$). The other disc dynamo is denoted j. Each disc dynamo i is made up of an axis of rotation, a disc and a wire, each made from the same conductive materials. These three elements constitute a closed electrical circuit, Ci, the wire being in sliding contact with the disc's rim at one end and with the axis at the other end. Electromagnetic coupling between the two disc dynamos is made with the loop of each wire around the axis of the other disc dynamo. Both disc dynamos are considered to be far enough apart to disregard any other interaction. The circuit made up by each (i) disc's dynamo rim neighbourhood, in which the azimuthal current is concentrated, is called Di. The set of parameters and unknowns is listed below.

The system depends on 16 parameters. For each disc dynamo (i), a given mechanical torque $G_i/2\pi$ is applied from the outside onto the axis. The moment of inertia of each disc (i) is $J_i/(2\pi)^2$, and its friction coefficient is denoted by $k_i/(2\pi)^2$. The circuit Di has an electrical resistance $R_{\mathrm{D}i}$ and a self-inductance $L_{\mathrm{D}i}$. The circuit Ci has an electrical resistance $R_{\mathrm{C}i}$ and a self-inductance $L_{\mathrm{C}i}$. The mutual inductance between Ci and D_j is M_{ji}.

There are 10 system unknowns: currents $I_{\mathrm{C}i}$ in Ci and $I_{\mathrm{D}i}$ in Di, magnetic flux $\Phi_{\mathrm{D}i}$ through Di and $\Phi_{\mathrm{C}i}$ through Ci, angular velocity $2\pi\Omega_i$ of each disc (i). In order to have a complete mathematical formulation of the problem, 10 independent equations describing relations between the unknowns are needed.

(*b*) *Governing equations*

The electric field created by the magnetic field $\boldsymbol{B}_i.$, assumed to be vertical and axisymmetric on the surface of the disc (i), is written as

$$\boldsymbol{E}(M) = 2\pi\Omega_1\boldsymbol{B}_i\boldsymbol{r}, \tag{2.1}$$

at any point $M(\boldsymbol{r})$ of the disc, where $\boldsymbol{r}$ denotes the radial cylindrical coordinate. This leads to a potential difference between the axis and the rim of the disc:

$$\Delta V = \Omega_i\Phi_{\mathrm{D}i}, \tag{2.2}$$

with

$$\Phi_{\mathrm{D}i} = 2\pi\int_0^R B_i r\,\mathrm{d}r.$$

Therefore, the electrical equation for the circuit (i) is given by

$$\Omega_i\Phi_{\mathrm{D}i} = R_{\mathrm{C}i}I_{\mathrm{C}i} + \Phi_{\mathrm{C}i}, \tag{2.3}$$

and the electrical equation for Di is reduced to

$$R_{\mathrm{D}i}I_{\mathrm{D}i} = -\dot{\Phi}_{\mathrm{D}i}. \tag{2.4}$$

The magnetic flux through Ci is due on the one hand to its self-inductance and, on the other, to the addition of the induction from (D_j). It follows that ($i \neq j$)

$$\Phi_{\mathrm{C}i} = L_{\mathrm{C}i}I_{\mathrm{C}i} + M_{ji}I_{D_j}, \tag{2.5}$$

and in a similar manner,

$$\Phi_{\mathrm{D}i} = L_{\mathrm{D}i}I_{\mathrm{D}i} + M_{ij}I_{C_j}. \tag{2.6}$$

The mechanical equation of the disc (i) is written

$$\frac{J_i}{(2\pi)^2}\frac{\mathrm{d}(2\pi\Omega_i)}{\mathrm{d}t} = \sum \text{moments applied to the disc } (i). \tag{2.7}$$

The moments applied to the disc (i) are composed of the torque $G_i/2\pi$, the moment of the mechanical friction $-\Omega_i k_i/2\pi$ and the moment of the Lorentz force. The Lorentz force is given by

$$\mathrm{d}^2\boldsymbol{F}_{\mathrm{Lor}} = \mathrm{d}I_{\mathrm{C}i}\mathrm{d}\boldsymbol{l} \times \boldsymbol{B}_i. \tag{2.8}$$

Thus, the moment of the Lorentz force is given by

$$M^t_{\mathrm{Lor}} = -I_{\mathrm{C}i}\frac{\Phi_{\mathrm{D}i}}{2\pi}. \tag{2.9}$$

Therefore, (2.7) is written

$$J_i\dot{\Omega}_i = G_i - \Phi_{\mathrm{D}i}I_{\mathrm{C}i} - k_i\Omega_i. \tag{2.10}$$

Consequently, equations (2.3)–(2.6) and (2.10) describe completely the system with 10 equations and 10 unknowns.

(*c*) *Dimensionless modelling*

The analysis of § 3 will be restricted to interactions between two identical discs. The discussion is then reduced to four non-dimensional parameters. Electrical currents are expressed in terms of magnetic flux, reducing the model to six unknowns and six independent equations. They are expressed in dimensionless form.

From (2.5) and (2.6), $I_{\mathrm{C}i}$ and I_{D_j} can be expressed in terms of $\phi_{\mathrm{C}i}$ and ϕ_{D_j}, and replaced in (2.4), (2.3) and (2.10). By defining the following dimensionless variables:

$$\tau = \frac{R_{\mathrm{C}}}{L_{\mathrm{C}}}t, \quad X_i = \frac{\Phi_{\mathrm{D}i}}{\sqrt{GM}}, \quad Y_i = \frac{\Phi_{\mathrm{C}i}}{L_{\mathrm{C}}\sqrt{G/M}}, \quad Z_i = \frac{M}{R_{\mathrm{C}}}\Omega_i, \tag{2.11}$$

the equations describing the behaviour of the two-disc dynamo are deduced from (2.4), (2.3) and (2.10) in the following form:

$$\left.\begin{aligned}
\dot{X}_i &= r(Y_j - X_i),\\
\dot{Y}_i &= X_iZ_i + mX_j - (1+m)Y_i,\\
\dot{Z}_i &= g\{1-(1+m)X_iY_i + mX_iX_j\} - fZ_i,
\end{aligned}\right\} \tag{2.12}$$

where

$$m = \frac{M^2}{L_{\mathrm{C}}L_{\mathrm{D}} - M^2}, \quad g = \frac{G}{J}\frac{ML_{\mathrm{C}}}{R_{\mathrm{C}}^2}, \quad r = \frac{R_{\mathrm{D}}}{R_{\mathrm{C}}}\frac{L_{\mathrm{C}}^2}{L_{\mathrm{C}}L_{\mathrm{D}} - M^2}, \quad f = \frac{kL_{\mathrm{C}}}{JR_{\mathrm{C}}}. \tag{2.13}$$

Since $M^2 < L_{\mathrm{C}}L_{\mathrm{D}}$, note that the magnetic interaction parameter m is always positive.

3. Analysis of the two-disc dynamo

(*a*) *The steady-state*

The steady-state solutions of the system (2.12) are given by

$$\begin{aligned}
X_1 &= Y_2, & X_2 &= Y_1,\\
X_1Z_1 &= Y_1, & X_2Z_2 &= Y_2,\\
g\{1 - X_1Y_1\} &= fZ_1, & g\{1 - X_2Y_2\} &= fZ_2.
\end{aligned} \tag{3.1}$$

For a coefficient f different from zero, (3.1) implies

$$Z_1 = Z_2, \quad X_1(Z_1^2 - 1) = X_2(Z_2^2 - 1) = 0. \tag{3.2}$$

Then two sets of steady-state solutions are obtained:

$$X_1 = X_2 = Y_1 = Y_2 = 0, \quad Z_1 = Z_2 = g/f, \tag{3.3 a}$$

and

$$X_1 = X_2 = Y_1 = Y_2 = \pm\sqrt{1-(f/g)}, \quad Z_1 = Z_2 = 1, \tag{3.3 b}$$

(3.3 *b*) making sense when $f \leqslant g$.

The solution of (3.2), $Z_i = -1$, leads to $g(1 + X_i^2) + f = 0$ and, consequently, is not acceptable in the non-trivial case.

The steady-state solutions X_i and Y_i are denoted by $X_{\rm st}$ and the Z_i are denoted by $Z_{\rm st}$. A linear stability analysis of each set of steady-state solutions (3.3 *a*) and (3.3 *b*) is proposed in the next section.

(*b*) *The linear stability analysis*

The stability of each steady-state solution set is investigated by adding to them small perturbations (x_i, y_i, z_i). Neglecting the quadratic terms, these perturbations verify the following system:

$$\left.\begin{aligned} \dot{x}_i &= r(y_j - x_i), \\ \dot{y}_i &= Z_{\rm st}x_i + X_{\rm st}z_i + mx_j - (1+m)y_i, \\ \dot{z}_i &= gX_{\rm st}(-x_i - (1+m)y_i + mx_j) - fz_i. \end{aligned}\right\} \tag{3.4}$$

Looking for non-trivial solutions of (3.4) proportional to $\exp(pt)$ where p is complex, leads to a polynomial function $\Delta(p)$ of sixth degree. The stability of the steady state is then given by the sign of the real part of the roots of $\Delta(p)$.

For $X_{\rm st} = 0$ and $Z_{\rm st} = g/f$, the polynomial function is given by

$$\begin{aligned} \Delta(p) = (f+p)^2[p^2 + (m+r+1)p + r(1+(g/f))] \\ \times [p^2 + (m+r+1)p + r(1-(g/f))]. \end{aligned} \tag{3.5}$$

The roots of Δ always have a negative real part, unless $r(1-(g/f)) < 0$. Therefore the steady state (3.3 *a*) is stable if and only if $g/f < 1$.

For $X_{\rm st} = \pm(1-(f/g))^{1/2}$ and $Z_{\rm st} = 1$, the polynomial function is given by

$$\begin{aligned} \Delta(p) = [p^3 + (m+r+f+1)p^2 + (g(m+1)+rf)p + 2r(g-f)] \\ \times [p^3 + (m+r+f+1)p^2 + (g(m+1)+r(f+2))p + 2rf]. \end{aligned} \tag{3.6}$$

The second factor of (3.6) has only negative real part roots (unless $rf = 0$). Indeed, it would lead to instability if and only if

$$\frac{2rf}{r+f+m+1} > g(m+1) + r(f+2) \Leftrightarrow -\frac{2r(r+m+1)}{r+f+m+1} > g(m+1) + rf,$$

which is not possible in a non-trivial case.

Therefore, the condition equivalent to linear instability of the solutions of (3.3) is

$$\frac{2r(g-f)}{r+f+m+1} > g(m+1) + rf, \tag{3.7}$$

or equivalently,

$$r > \frac{(m+1)(m+f+1)}{1-m} \tag{3.8 a}$$

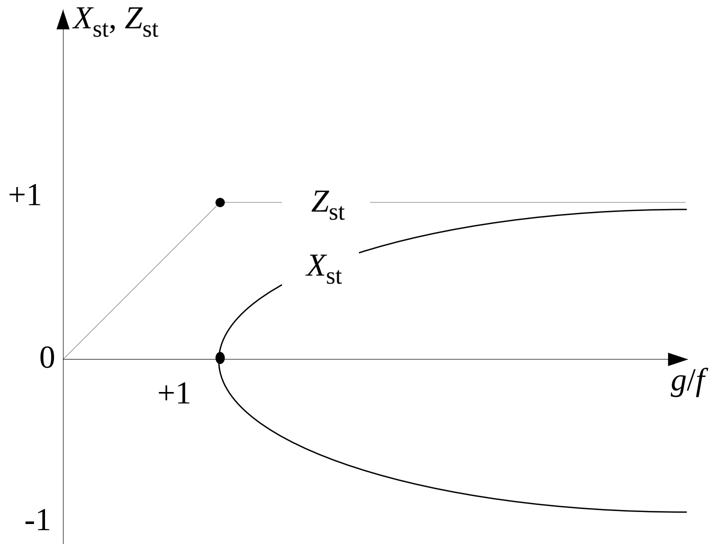

Figure 2. Bifurcation diagram of the two-disc dynamo model. Assuming that condition (3.8 a) is verified, then a stable steady-state solution ($X_{st}; Z_{st}$) exists unless g/f is sufficiently large to verify (3.8 b).

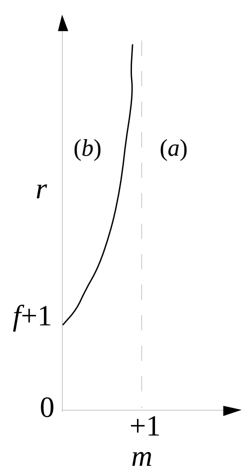

Figure 3. Illustration of the linear stability analysis of the steady-state $X_{st} = \pm(1 - f/g)^{1/2}$ and $Z_{st} = 1$. (a) Always stable. (b) Stable unless (3.8 b) is verified.

and

$$\frac{g}{f} > \frac{2r + r(r + f + m + 1)}{2r - (m + 1)(r + m + f + 1)}. \tag{3.8 b}$$

In figure 2, the stable steady-state solutions of the system (2.12) are plotted in terms of g/f. It is interesting to note that condition (3.8 a) only concerns the parameters r, m and f. In figure 3, the region of the (r, m)-plane corresponding to linear instability (3.8 a) is indicated.

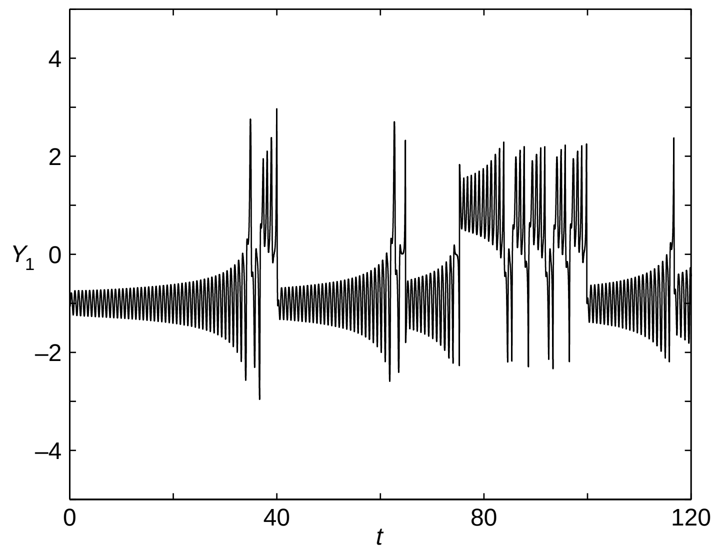

Figure 4. A numerical computation of Y_1 versus time, with $(m; g; r; f) = (0.5; 50; 8; 0, 5)$, $(X_1; X_2; Y_1; Y_2; Z_1; Z_2)_{\text{initial}} = (-1.4; -1; -1; -1.4; 2.2; -1.5)$.

One notes that, unless on bifurcation points or on the boundary between stability and instability, the roots of $\Delta(p)$ do not have zero real part. So, from the Lyapunov theorems, stability of the nonlinear initial system (2.12) can be deduced from the results of the linear stability analysis.

In figure 4, a numerical computation of Y_1 versus time is presented for parameters verifying (3.7). The behaviour of the solution is chaotic and presents typical reversals.

4. Formulation and analysis of the one-disc dynamo

The dimensionless model for the one-disc dynamo with azimuthal current and mechanical friction can be deduced from (2.12), by setting $X_i = X_j = X$, $Y_i = Y_j = Y$ and $Z_i = Z_j = Z$:

$$\left.\begin{aligned} \dot{X} &= r(Y - X), \\ \dot{Y} &= XZ + mX - (1+m)Y, \\ \dot{Z} &= g\{1 - (1+m)XY + mX^2\} - fZ. \end{aligned}\right\} \tag{4.1}$$

The steady-state solutions of (4.1) are identical to (3.3 a) and (3.3 b). The linear stability analysis of (4.1) also leads to the same condition of instability (3.7) and bifurcation diagram (figure 2) as for the two-disc dynamo. Therefore, the conclusions concerning the two-disc dynamo are also valid for the one-disc dynamo.

5. Conclusion

The two-disc dynamo has been studied taking into account the existence of azimuthal currents in the discs and non-zero mechanical friction. The linear stability of the steady solutions of the symmetric system (identical discs) has been analysed and has shown that both stable and unstable cases are expected, depending on four parameters derived from the dimensionless equations. In particular, unstable steady solutions are possible when these parameters verify two specific conditions (3.8 *a*) and (3.8 *b*). If (3.8 *a*) is verified, then (3.8 *b*) shows that there is always a mechanical torque G applied to the discs, sufficiently large to lead to instabilities. The existence of these instabilities is probably due to the existence of azimuthal currents. Indeed, they have the possibility of amplifying the magnetic field induced by the electrical loop, overbalancing the damping induced by mechanical friction.

In the case where azimuthal currents are neglected ($1/r = 0$), then (3.8 *b*) is not verified. This confirms that the two-disc dynamo system as formulated by Rikitake but including symmetric mechanical friction, is in fact always stable, as emphasized by Hide (1995). In the case where friction coefficients are neglected ($f = 0$), then the system is structurally unstable and so not physically realistic. Therefore, this analysis proves that it is not possible to neglect either the azimuthal currents or mechanical friction without any loss of generality. Indeed, such a false analysis would either lead to over stability or to structural instability. Moreover, taking into account these two important features (azimuthal currents and mechanical frictions), chaotic solutions have been found, depending on the parameters of the problem. Therefore, the two-disc dynamo as modelled by Rikitake but incorporating symmetric mechanical friction and azimuthal currents, stays, to our knowledge, the simplest system presenting magnetic field reversals.

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