RETRIEVAL OF THE GREEN FUNCTION FROM CROSS-CORRELATION: THE CANONICAL ELASTIC PROBLEM

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ABSTRACT

In realistic materials, multiple scattering takes place and average field intensities or energy densities follow diffusive processes. Multiple P to S energy conversions by the random inhomogeneities result in equipartition of elastic waves which means that, in the phase space the available elastic energy is distributed among all the possible states of P and S waves, with equal amounts in average. In such diffusive regimes the P to S energy ratio equilibrates in a universal way *independent of the particular details* of the scattering. We study the canonical problem of isotropic plane waves in an elastic medium and show that the Fourier transform of azimuthal average of the cross-correlation of motion between two points within an elastic medium is proportional to the imaginary part of the exact Green tensor function between these points, provided the energy ratio E_S/E_P is the one predicted by equipartition in two- and three-dimensions, respectively. These results clearly show that equipartition is a necessary condition to retrieve the *exact* Green function from correlations of the elastic field. However, even if there is not an equipartitioned regime and correlations do not allow to retrieve precisely the exact Green function they may provide valuable results of physical significance by reconstructing specific arrivals.

INTRODUCTION

In order to build descriptions of the Earth structure useful for exploration or seismic risk studies, geophysicists have studied seismic noise and the coda waves generated after the passage of the direct arrivals, also called ballistic waves. In his pioneering studies Aki pointed out that seismic noise and coda waves may contain valuable information of the propagation media (Aki, 1957; Aki and Chouet, 1975). Aki's long term program ranged from single and multiple scattering descriptions to radiative transfer ideas aimed to explain coda envelopes. A comprehensive account can be found in Sato and Fehler (1998).

It has been demonstrated recently that the elastodynamic Green function can be recovered from isotropic elastic wavefield generated by either multiple scattering or by a large number of sources (such as microseisms) as well (Campillo and Paul, 2003; Shapiro and Campillo, 2004, Sabra *et al*., 2005). Moreover, these experimental results clearly show the existence of long range correlation.

On the other hand, equipartition means that in the phase space the available energy is equally distributed, with fixed average amounts, among all the possible states. This simple, yet powerful concept is one of the building blocks of modern thermodynamics, has also been put forward for multiple scattered waves and specifically for elastic waves. To establish the corresponding ratios several devices have been used. In fact, for waves within a diffusive regime, Weaver (1982) employed an elegant mode counting argument that we explain in the Appendix together with an interpretation in the phase space. Later he showed (Weaver, 1990) the equipartition as a limit of multiple scattering whereas Ryzhik *et al.* (1996) formally established the transport equation of elastic waves and the associated diffusion approximation. They pointed out that in such diffusive regimes the P to S energy ratio equilibrates in a universal way *independent of the details* of the multiple scattering. On the other hand, Snieder (2002) used an ingenious probabilistic ball-counting algorithm.

In real materials the multiple scattering that takes place generates an interesting fact. Although in the micro scale the field equations remain the same (Newton and Hooke's laws, thus Navier's equations) intensities, like other averages, follow diffusive processes and these averages satisfy diffusion-like equations (*e.g.* heat equation). Therefore, in the diffusive regime (when traveled distances are much longer than the transport mean free path and the travel times long compared to the transport mean free time) the P to S energy conversion by the random inhomogeneities equilibrates in a universal way *independent of the details* of the scattering. The equipartition of energy in 3D for an homogeneous background medium leads to the relation $E_S(t,\mathbf{x}) = 2(\alpha/\beta)^3 E_P(t,\mathbf{x})$, where E_S and E_P are the S and P spatial energy densities, and α and β are the P and S wave speeds, respectively.

Evidence of the transition toward equipartition has been observed in real data (Shapiro *et al.*, 2000, and Hennino *et al.*, 2001) from the observation of the stabilization of the P to S energies ratio in the coda at a value compatible with the theory. In fact, coda waves are natural candidates to undergo equipartition. This is attested by the stabilization of the energies ratio. Coda waves continue ringing long for a duration which is various times the source-site travel time and this suggests, at least, intense multipathing. The exponential decay of coda waves, characterized by the coda Q which, although frequency dependent, is a regional constant, independent of magnitude and source depth (Aki and Chouet, 1975). This is very strong indication that coda waves sample the Earth uniformly around the recording station. Equipartition is expected to arise naturally in the diffusive regime. The energy ratio stabilization appears before the complete isotropy of the field but indicates that the diffusion can be used as a good approximation (Paul *et al*., 2005).

In this work we deal with the canonical problem of a uniform random distribution of plane waves within a homogeneous elastic medium. The cross correlation of the fields produced at two points by generic plane waves is computed, then azimuthally averaged. Polar and spherical coordinates are used for 2D and 3D, respectively. We show that the Fourier transform of the average of the cross-correlation of motion between two points is proportional to the imaginary part of the tensor Green function between these points, provided the energy ratio E_S/E_P is $(\alpha/\beta)^2$ and $2(\alpha/\beta)^3$, in two- and tree-dimensions, respectively (see the Appendix). These results clearly show that, for an elastic medium, equipartition is a necessary condition to retrieve the Green function from correlations of the isotropic elastic field. However, the usefulness of correlations is not confined to the equipartitioned case. Indeed, correlations do provide significant information even in cases where the fully diffuse nature of the fields is not at all obvious as it was demonstrated in the several applications already mentioned.

The case of the homogeneous, isotropic, elastic body with an isotropic random distribution of plane waves is an important canonical problem. We retrieve the exact properties of Green function, like distance behavior of P and S waves and the precise balance between P and S amplitudes. Obviously, the energy ratio between P and S in the diffuse incoming random field governs the balance in the correlation. In the elastic case the particular value of this ratio that leads to the exact full Green function is precisely the one predicted by the theory of Equipartition. We also discuss the implications of these results for the retrieval of the surface terms of Green function in layered medium.

THE 2D SCALAR CASE

Let's start with a two-dimensional scalar field. Without loss of generality we assume we are dealing with *SH* waves in a homogeneous elastic medium (see *e.g*. Aki and Richards, 1980). Propagation takes place in the x_1-x_3 plane. Therefore, the antiplane (out-of-plane) displacement $v(\mathbf{x},t)$ fulfils the wave equation

$$\frac{\partial^2 v}{\partial x_1^2}+\frac{\partial^2 v}{\partial x_3^2}=\frac{1}{\beta^2}\frac{\partial^2 v}{\partial t^2}, \tag{1}$$

where β = shear wave velocity and t = time. A typical harmonic, homogeneous plane wave can be written as

$$v(\mathbf{x},\omega,t)=F(\omega,\psi)\exp(-\mathrm{i}kx_jn_j)\exp(\mathrm{i}\omega t), \tag{2}$$

where $k=\omega/\beta$ = S wavenumber , $F(\omega,\psi)$ = complex waveform, ω=circular frequency, $\mathbf{x}^{\mathrm{T}} = (x_1, x_3)$ = Cartesian coordinates (such that $x_1 = r\cos\theta = r\gamma_1$, $x_3 = r\sin\theta = r\gamma_3$, with r, θ = polar coordinates) and n_j = direction cosines (n_1= cos ψ , n_3 = sin ψ) that define wave propagation.

Consider the correlation of the motion described in Eq. 2, evaluated at positions $\mathbf{x}$ and $\mathbf{y}$, respectively. For simplicity assume $\mathbf{y}$ at the origin. In this way the scalar product $n_jx_j = rn_j\gamma_j = r\cos[\psi-\theta]$. Thus, we can write

$$v(\mathbf{y},\omega)v^*(\mathbf{x},\omega)=F(\omega,\psi)F^*(\omega,\psi)\exp(\mathbf{i}kr\cos[\psi-\theta]). \tag{3}$$

Here and hereafter the time factor $\exp(\mathrm{i}\omega t)$ is omitted. If we assume an isotropic field in which waves propagate back and forth in directions given by ψ and such that the average spectral density is $F(\omega,\psi)F^*(\omega,\psi) = |F(\omega)|^2$, roughly independent of ψ, then an azimuthal average over ψ leads to

$$\langle v(\mathbf{y},\omega)v^*(\mathbf{x},\omega)\rangle = |F(\omega)|^2 \frac{1}{2\pi}\int_0^{2\pi} \exp(\mathbf{i}kr\cos[\psi-\theta])d\psi = |F(\omega)|^2 J_0(kr), \quad (4)$$

where $J_0(kr)$ = Bessel function of the first kind and zero order of argument kr. This result comes out naturally from the Neumann expansion of the complex exponential

$$\exp(\mathrm{i}kr\cos[\psi-\theta]) = \sum_{n=0}^{\infty} \varepsilon_n \mathrm{i}^n J_n(kr)\cos n[\psi-\theta], \quad (5)$$

where ε_n= Neumann factor (=1 if n=0, =2 for n>0). It is clear that the only contribution to the integral comes from n=0. The result in Eqn. 4 has been found by Aki (1957) in the framework of his study of microtremors. It is the basis of the Spectral Auto-Correlation (SPAC) method.

In acoustic problems, the average correlations between a pair of sensors placed within isotropic noise lead to the zero order Bessel functions, cylindrical or spherical, in two- and three-dimensions, respectively, as was well known since the early 60's. In his classic paper Cox (1973) overcame the isotropy assumption and considered an arbitrary directional distribution of uncorrelated plane waves. He expanded such distribution using spatial harmonics and obtained analytical expressions for each term of the expansion. It is possible to study elastic fields using Cox (1973) approach.

Consider now the Green function in the frequency domain

$$G_{22}(\mathbf{x}, \mathbf{y}; \omega) = 1/4\mathrm{i}\mu\, [J_0(kr) - \mathrm{i}Y_0(kr)], \quad (6)$$

where $Y_0(kr)$ = Neumann function of zero order and μ = shear modulus. From Eqs. 4 and 6 it is clear that $J_0(kr)$ is proportional to the imaginary part of the Green function. In fact, if $E_{SH} = \rho\omega^2 |F(\omega)|^2 / 2$ = energy density for SH waves, we can write

$$\langle v(\mathbf{y},\omega)v^*(\mathbf{x},\omega)\rangle = \frac{2}{\rho\omega^2} E_{SH} J_0(kr) = -8E_{SH}k^{-2}\,\mathrm{Im}[G_{22}(\mathbf{x},\mathbf{y},\omega)], \quad (7)$$

It is convenient to establish that $J_0(kr)$ contains all the information regarding the Green function. Its real part is just the Hilbert transform of the imaginary one ($J_0(kr)$ and $Y_0(kr)$ form a Hilbert pair. See, *e. g.* Aki and Richards,1980 for a discussion). Therefore, the transform of $-\mathrm{i}J_0(kr)$ gives a signal that is proportional to the causal Green function with an even contribution in the negative times.

More precisely, taking the inverse Fourier transformation of Eq. 6, we have

$$G_{22}(\mathbf{x},\mathbf{y},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G_{22}(\mathbf{x},\mathbf{y},\omega)\exp(\mathbf{i}\omega t)d\omega = \frac{1}{2\pi\mu}\frac{H(t-r/\beta)}{\sqrt{t^2-r^2/\beta^2}}, \quad (8)$$

where H = Heaviside function. We should note that this is a causal Green function because *imaginary and real part contribute equally in the positive times and cancel out exactly in negative times*.

THE 2D VECTOR CASE

With reference to Fig. 1, assume we are dealing with P and SV waves in a homogeneous, isotropic, elastic medium. Again, propagation takes place in the x_1-x_3 (or x-z) plane. Therefore, the in-plane displacements $u_i(\mathbf{x},t)$, where i=1,3, fulfills Navier equation

$$\beta^2 \frac{\partial^2 u_i}{\partial x_j \partial x_j} + (\alpha^2 - \beta^2)\frac{\partial^2 u_j}{\partial x_i \partial x_j} = \frac{\partial^2 u_i}{\partial t^2}, \tag{9}$$

where α = compressional wave velocity and t = time. In Eq. 9 the Einstein summation convention is used. Let us remember the form of the Green's function (*e. g.* Sánchez-Sesma and Campillo, 1991):

$$G_{ij}(\mathbf{x},\mathbf{y};\omega) = \frac{1}{\mathrm{i}8\rho}\left\{A\delta_{ij} - B(2\gamma_i\gamma_j - \delta_{ij})\right\} \quad i,j=1,3, \tag{10}$$

where ρ = mass density,

$$A = \frac{H_0^{(2)}(qr)}{\alpha^2} + \frac{H_0^{(2)}(kr)}{\beta^2} \text{ and } B = \frac{H_2^{(2)}(qr)}{\alpha^2} - \frac{H_2^{(2)}(kr)}{\beta^2}, \tag{11}$$

with $H_m^{(2)}(\bullet) = J_m(\bullet) - \mathrm{i}Y_m(\bullet)$ = Hankel function of the second kind and order m expressed in terms of the Bessel functions of the first and second kind. The S and P wavenumbers are given by $k = \omega/\beta$ and $q = \omega/\alpha$, respectively.

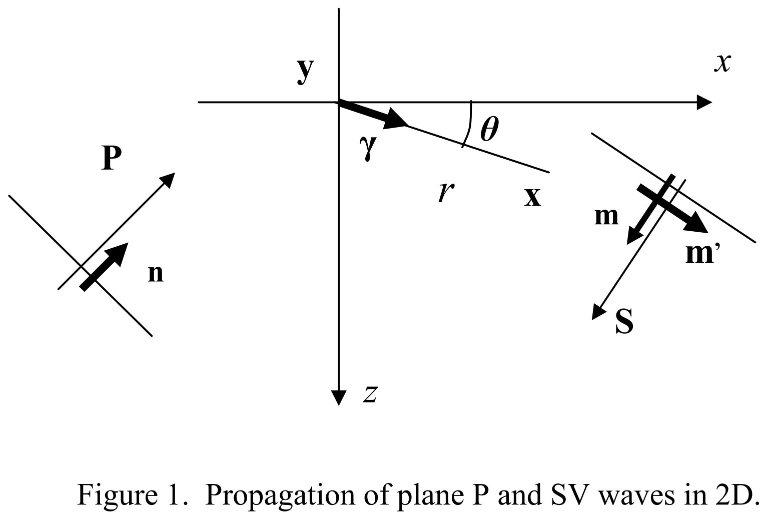

Figure 1. Propagation of plane P and SV waves in 2D.

The propagation of both P and SV waves assuming the field composed of harmonic, homogeneous plane waves can be described by means of

$$u_l(\mathbf{x},\omega,t) = P(\omega,\varphi)n_l \exp(-\mathrm{i}qx_jn_j) + S(\omega,\psi)m_l^{'} \exp(-\mathrm{i}kx_jm_j)\,, \tag{12}$$

where, P and S = complex waveforms and m_j , n_j = direction cosines (m_1=cosψ , m_3 = sinψ, n_1= cos φ , n_3 = sin φ). Note that polarization of the P part is given by n_l while for S part we have $m_1^{'} = -m_3$ and $m_3^{'} = m_1$ and this guarantees the proper polarization for shear waves (see *e. g.* Aki and Richards, 1980).

Consider the cross-correlation of this vector motion, evaluated at positions **x** and **y**, respectively. For simplicity assume **y** at the origin. Thus, we can write

$$\begin{aligned} u_l(\mathbf{y})u_s^*(\mathbf{x}) &= (P^2 n_l n_s + SP^* m_l^{'} n_s)\exp(\mathbf{i}qr\cos[\varphi-\theta]) \\ &\quad + (S^2 m_l^{'} m_s^{'} + PS^* n_l m_s^{'})\exp(\mathbf{i}kr\cos[\psi-\theta]) \end{aligned} \tag{13}$$

Specializing Eq. 11 for l=1 and s=1 we have

$$\begin{aligned} u_1(\mathbf{y})u_1^*(\mathbf{x}) &= (P^2\cos^2\varphi - SP^*\sin\psi\cos\varphi)\exp(\mathbf{i}qr\cos[\varphi-\theta]) \\ &\quad + (S^2\sin^2\psi - PS^*\cos\varphi\sin\psi)\exp(\mathbf{i}kr\cos[\psi-\theta]) \end{aligned} \tag{14}$$

for l=3 and s=3 we can write

$$\begin{aligned} u_3(\mathbf{y})u_3^*(\mathbf{x}) &= (P^2\sin^2\varphi + SP^*\cos\psi\sin\varphi)\exp(\mathbf{i}qr\cos[\varphi-\theta]) \\ &\quad + (S^2\cos^2\psi + PS^*\sin\varphi\cos\psi)\exp(\mathbf{i}kr\cos[\psi-\theta]) \end{aligned} \tag{15}$$

and finally for l=1 and s=3 we have

$$\begin{aligned} u_1(\mathbf{y})u_3^*(\mathbf{x}) &= (P^2\cos\varphi\sin\varphi - SP^*\sin\psi\sin\varphi)\exp(\mathbf{i}qr\cos[\varphi-\theta]) \\ &\quad + (-S^2\sin\psi\cos\psi + PS^*\cos\varphi\cos\psi)\exp(\mathbf{i}kr\cos[\psi-\theta]) \end{aligned} \tag{16}$$

Assume an isotropic 2D field in which P and SV waves propagate in the directions given by φ and ψ. Spectral densities P^2 and S^2 are independent of the propagation angles. Let us assume further that $P^2\alpha^2 = \varepsilon\, S^2\beta^2$. Then the azimuthal average over φ and ψ of Eqs 14-16, taking into account Eq. 5, leads to

$$\left\langle u_i(\mathbf{y})u_j^*(\mathbf{x})\right\rangle = \frac{S^2\beta^2}{2}\left\{A\delta_{ij} - B(2\gamma_i\gamma_j - \delta_{ij})\right\}, \tag{17}$$

where

$$A = \varepsilon\frac{J_0(qr)}{\alpha^2} + \frac{J_0(kr)}{\beta^2} \text{ and } B = \varepsilon\frac{J_2(qr)}{\alpha^2} - \frac{J_2(kr)}{\beta^2}\,, \tag{18}$$

with $J_n(kr)$ = Bessel function of the first kind and order *n* of argument *kr*. This result is precisely the extension of the scalar SH case. It can be demonstrated that the cross terms cancel out in the averaging. If $\varepsilon = 1$ in Eqs. 18 then from Eqs. 10 and 11 one can see that the expression within brackets in Eq. 17 is precisely $-8\rho\,\mathrm{Im}[G_{ij}(\mathbf{x},\mathbf{y};\omega)]$. Considering that $\mu = \rho\beta^2$ and $E_S = \rho\omega^2 S^2/2$ we can write

$$\left\langle u_i(\mathbf{y},\omega)u_j^*(\mathbf{x},\omega)\right\rangle = -8E_S k^{-2}\,\mathrm{Im}\left[G_{ij}(\mathbf{x},\mathbf{y},\omega)\right], \tag{19}$$

which generalizes the result of Eq. 7. These two equations are remarkable. It can be shown that they hold if **x** and **y** and/or *i* and *j* are exchanged. These properties are consequences of reciprocity.

The energy ratio we consider ($\varepsilon = 1$) can be interpreted as a 2D expression of the energy Equipartition Principle in the only two states of propagation, namely P and S waves, that is easily found to be $E_S/E_P = (\alpha/\beta)^2$ by a mode counting argument (see Appendix).

The case without equipartition, namely $P^2\alpha^2 \neq S^2\beta^2$, does not allow to retrieve the exact Green function. However, it is of great practical interest as all the Green function components are there, with amplitudes for each wave consistent with the actual energy ratio.

THE 3D VECTOR CASE

Now we have to deal with P, SV and SH waves in a homogeneous, isotropic, elastic medium. Propagation takes place in the 3D and displacements $u_i(\mathbf{x},t)$, where i=1, 2, and 3, fulfills again Navier's equation:

$$\beta^2\frac{\partial^2 u_i}{\partial x_j \partial x_j} + (\alpha^2-\beta^2)\frac{\partial^2 u_j}{\partial x_i \partial x_j} = \frac{\partial^2 u_i}{\partial t^2}, \tag{20}$$

where both *i* and *j* run from 1 to 3.

It can be shown (Sánchez-Sesma and Luzón, 1995) that the Green tensor is given by

$$G_{ij}(\mathbf{x},\mathbf{y};\omega) = \frac{1}{4\pi\mu r}\left[f_2\delta_{ij} + (f_1 - f_2)\gamma_i\gamma_j\right], \tag{21}$$

where f_1 and f_2 are Stokes' functions, and they are given by

$$\begin{aligned} f_1 &= (\beta/\alpha)^2\left[1 - 2\mathrm{i}/(qr) - 2/(qr)^2\right]\exp(-\mathrm{i}qr) + \left[2\mathrm{i}/(kr) + 2/(kr)^2\right]\exp(-\mathrm{i}kr), \text{ and} \\ f_2 &= (\beta/\alpha)^2\left[\mathrm{i}/(qr) + 1/(qr)^2\right]\exp(-\mathrm{i}qr) + \left[1 - \mathrm{i}/(kr) - 1/(kr)^2\right]\exp(-\mathrm{i}kr). \end{aligned} \tag{22}$$

These functions have the constants 1 and $\left(1+(\beta/\alpha)^2\right)/2$, respectively as limits if ω or r tend to zero. It is convenient for our analysis to express f_1 and f_2 in terms of spherical Hankel's functions of the second kind. Then, it is possible to show that

$$f_1 = -\mathrm{i}kr\beta^3(a-2b)/3, \qquad f_2 = -\mathrm{i}kr\beta^3(a+b)/3, \tag{23}$$

where

$$a = \frac{h_0^{(2)}(qr)}{\alpha^3} + \frac{2h_0^{(2)}(kr)}{\beta^3}, \qquad b = \frac{h_2^{(2)}(qr)}{\alpha^3} - \frac{h_2^{(2)}(kr)}{\beta^3}, \text{ and} \tag{24}$$

$h_m^{(2)}(z) = j_m(z) - \mathrm{i}y_m(z) =$ spherical Hankel's function of the second kind and order m expressed in terms of the spherical Bessel functions of the first and second kind, respectively. The structure of Eqs. 24 is similar to that of Eqs. 11, in the 2D elastic case, the differences are in the cubes of propagation velocities in denominators and the factor 2 in the second term of a.

Consider the propagation of P, SV and SH waves (see Fig. 2) assuming the field composed of harmonic, homogeneous plane waves by means of

$$\begin{aligned} u_l(\mathbf{x},\omega,t) = {} & P(\omega,\theta_0,\varphi_0)n_l \exp(-\mathrm{i}qx_j n_j) \\ & + S_V(\omega,\theta_1,\varphi_1)m_l' \exp(-\mathrm{i}kx_j m_j) \quad , \\ & + S_H(\omega,\theta_2,\varphi_2)h_l' \exp(-\mathrm{i}kx_j h_j) \end{aligned} \tag{25}$$

where, $k=\omega/\beta$ and $q=\omega/\alpha$, P, S_V, and S_H = complex waveforms and n_j, m_j, h_j = direction cosines, respectively.

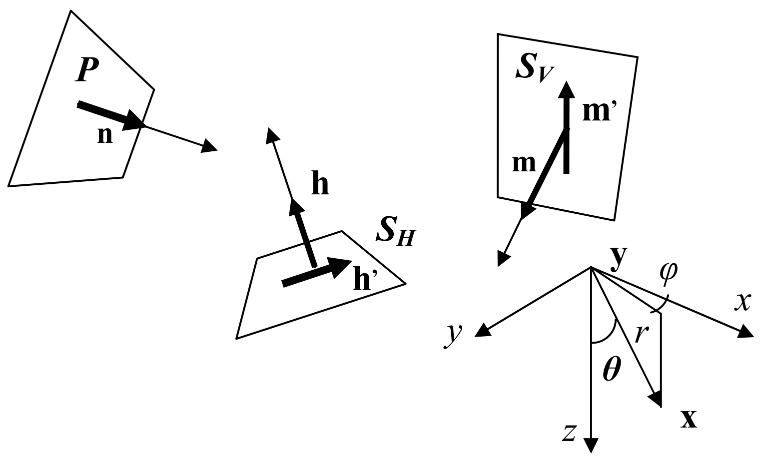

Figure 2. Propagation of plane P, SV and SH waves in 3D.

A convenient way to express all the possible propagation directions is by means of the spherical coordinates r, θ , and φ (the values j =1, 2 and 3 correspond as usual to Cartesian coordinates):

$$\{n_j\} = \begin{Bmatrix} \sin\theta_0 \cos\varphi_0 \\ \sin\theta_0 \sin\varphi_0 \\ \cos\theta_0 \end{Bmatrix}, \quad \{m_j\} = \begin{Bmatrix} \sin\theta_1 \cos\varphi_1 \\ \sin\theta_1 \sin\varphi_1 \\ \cos\theta_1 \end{Bmatrix}, \quad \text{and} \quad \{h_j\} = \begin{Bmatrix} \sin\theta_2 \cos\varphi_2 \\ \sin\theta_2 \sin\varphi_2 \\ \cos\theta_2 \end{Bmatrix}. \tag{26}$$

The polarization of motions is important and has to be described using the spherical coordinates. P waves are polarized as the propagation directions n_j . For SV and SH waves we choose vertical and horizontal polarizations expressed by

$$\{m'_j\} = \begin{Bmatrix} \cos\theta_1 \cos\varphi_1 \\ \cos\theta_1 \sin\varphi_1 \\ -\sin\theta_1 \end{Bmatrix}, \quad \text{and} \quad \{h'_j\} = \begin{Bmatrix} -\sin\varphi_2 \\ \cos\varphi_2 \\ 0 \end{Bmatrix}. \tag{27}$$

It is convenient to express the position vector x_j as $r\gamma_j$ where

$$\{\gamma_j\} = \begin{Bmatrix} \sin\theta \cos\varphi \\ \sin\theta \sin\varphi \\ \cos\theta \end{Bmatrix}. \tag{28}$$

Consider now the cross-correlation of the vector motion, evaluated at positions **x** and **y**, respectively. For simplicity assume **y** at the origin and **x** at $\theta = 0$ with a distance r from the origin. Thus, we can write

$$\begin{aligned} u_l(\mathbf{y})u_s^*(\mathbf{x}) = {} & (P^2 n_l n_s + S_V P^* m'_l n_s + S_H P^* h'_l n_s) \exp(\mathbf{i}qr\cos\theta_0) \\ & + (PS_V^* n_l m'_s + S_V^2 m'_l m'_s + S_H S_V^* h'_l m'_s) \exp(\mathbf{i}kr\cos\theta_1) \\ & + (PS_H^* n_l h'_s + S_V S_H^* m'_l h'_s + S_H^2 h'_l h'_s) \exp(\mathbf{i}kr\cos\theta_2). \end{aligned} \tag{29}$$

This vector product is a second order tensor and is expressed as the sum of nine elementary tensors (the exponentials are scalars). The squares in this equation stand for spectral densities. Let's perform the azimuthal average over all possible incidences of P, SV and SH waves by formally applying three times (over the angles φ_m and θ_m for m=0, 1 and 2, respectively) upon our tensor of Eq. 29 of an operator of the form

$$\langle \bullet \rangle = \frac{1}{4\pi} \int_0^{2\pi} d\varphi \int_0^{\pi} \bullet \sin\theta \, d\theta, \tag{30}$$

which gives one when acting over a unitary constant because the integrand of the double integral is the differential of area of the unit sphere.
Performing the average for l=1 and s=1 one obtains

$$\left\langle u_1(\mathbf{y})u_1^*(\mathbf{x})\right\rangle = \frac{P^2}{4}\int_0^{\pi}\sin^2\theta_0 \exp(\mathrm{i}qr\cos\theta_0)\sin\theta_0 d\theta_0 + \frac{S_V^2}{4}\int_0^{\pi}\cos^2\theta_1 \exp(\mathrm{i}kr\cos\theta_1)\sin\theta_1 d\theta_1 + \frac{S_H^2}{4}\int_0^{\pi}\exp(\mathrm{i}kr\cos\theta_2)\sin\theta_2 d\theta_2, \qquad (31)$$

for l=2 and s=2 we find:

$$\left\langle u_2(\mathbf{y})u_2^*(\mathbf{x})\right\rangle = \left\langle u_1(\mathbf{y})u_1^*(\mathbf{x})\right\rangle. \qquad (32)$$

For l=3 and s=3 the result is

$$\left\langle u_3(\mathbf{y})u_3^*(\mathbf{x})\right\rangle = \frac{P^2}{2}\int_0^{\pi}\cos^2\theta_0 \exp(\mathbf{i}qr\cos\theta_0)\sin\theta_0 d\theta_0 + \frac{S_V^2}{2}\int_0^{\pi}\sin^2\theta_1 \exp(\mathbf{i}kr\cos\theta_1)\sin\theta_1 d\theta_1. \qquad (33)$$

It can be demonstrated that the other terms of the average correlation tensor $\left\langle u_l(\mathbf{y})u_s^*(\mathbf{x})\right\rangle$, *i. e*. those for $l \neq s$, cancel out.

Considering the Poisson-Gegenbauer’s Integral (see Abramowitz and Stegun, 1972)

$$j_n(z) = \frac{1}{2}(-\mathrm{i})^n \int_0^{\pi}\exp(\mathrm{i}z\cos\theta)P_n(\cos\theta)\sin\theta d\theta, \qquad (34)$$

where $P_n(\cos\theta) =$ Legendre polynomial of order n, and the identities $P_0(\cos\theta) = 1$, $P_1(\cos\theta) = \cos\theta$, and $P_2(\cos\theta) = (3\cos^2\theta - 1)/2$, then Eqs. 31-33 can be written as

$$\left\langle u_1(\mathbf{y})u_1^*(\mathbf{x})\right\rangle = \frac{P^2}{3}\left(j_0(qr) + j_2(qr)\right) + \frac{S_V^2}{6}\left(j_0(kr) - 2j_2(kr)\right) + \frac{S_H^2}{2}j_0(kr), \text{and} \qquad (35)$$

$$\left\langle u_3(\mathbf{y})u_3^*(\mathbf{x})\right\rangle = \frac{P^2}{3}\left(j_0(qr) - 2j_2(qr)\right) + \frac{2S_V^2}{3}\left(j_0(kr) + j_2(kr)\right). \qquad (36)$$

The spectral densities P^2, S_V^2, and S_H^2 are independent of propagation angles. Let us assume that they satisfy the relationship $P^2\alpha^3 = S_V^2\beta^3 = S_H^2\beta^3$ which express that, in average, the energy ratio of S (including SH and SV modes) to P waves is given by $E_S / E_P = 2\alpha^3/\beta^3$, the equipartition ratio for 3D elastic fields (see Appendix). Here $E_S = \rho\omega^2 S^2/2$, with $S^2 = S_V^2 + S_H^2$. Therefore, Eqs. 35-36 and the identity $\langle u_1(\mathbf{y})u_1^*(\mathbf{x})\rangle = \langle u_2(\mathbf{y})u_2^*(\mathbf{x})\rangle$, taking into account Eqs. 20, 23 and 24, lead to

$$\langle u_i(\mathbf{y},\omega)u_j^*(\mathbf{x},\omega)\rangle = -4\pi E_S k^{-3}\,\mathrm{Im}\left[G_{ij}(\mathbf{x},\mathbf{y};\omega)\right]. \tag{37}$$

Note that this result corresponds to $\theta = 0$ (or $\gamma_1 = \gamma_2 = 0$, $\gamma_3 = 1$) and then, from Eq. 20, we have $G_{11}(\mathbf{x},\mathbf{y};\omega) = f_2/4\pi\mu r$, $G_{22}(\mathbf{x},\mathbf{y};\omega) = f_2/4\pi\mu r$ and $G_{33}(\mathbf{x},\mathbf{y};\omega) = f_1/4\pi\mu r$, being null the other components of Green function. Because both the azimuthal average correlation and the Green function are tensors their equality remains valid for any reference system. Therefore, Eq. 37 is valid for arbitrary **x** and **y**. Again, note that Eq. 36 holds if **x** and **y** and/or *i* and *j* are exchanged, because of reciprocity.

Without equipartition, $P^2\alpha^2 \neq S_V^2\beta^2 \neq S_H^2\beta^2$, we do not retrieve precisely the exact Green function. However, the relevant physics is there with amplitudes for each wave consistent with the actual energy ratios.

Note that Eq. 34 for n=0 and $z=kr\cos\theta$ with $k=\omega/c(\omega)$ expresses the azimuthal average of correlation coefficient for the acoustic case in 3D.

DISCUSSION

The proportionality of the correlation and the Green function is established for a specific ratio between the energy of isotropically distributed P and S waves. This energy ratio is an expression of the Principle of Equipartition of energy in the states of propagation, namely P and S waves. The fulfillment of this condition is necessary for Eqs. 19 and 37 to be valid.

In practice the azimuthal average is replaced by the stacking of correlations of coda records from different earthquakes of various magnitudes. Normalization is therefore required and normalized versions of Eqs. 19 and 37 are more useful.

The density of energy E can be estimated in each realization as

$$E = \frac{1}{2}\rho\omega^2\left|\mathbf{u}(\mathbf{y},\omega)\right|^2 = \frac{1}{2}\rho\omega^2\left|\mathbf{u}(\mathbf{x},\omega)\right|^2 \approx \frac{1}{2}\rho\omega^2\left|\mathbf{u}(\mathbf{x},\omega)\right| \times \left|\mathbf{u}(\mathbf{y},\omega)\right|. \tag{38}$$

Using the equipartition condition, the total energy densities in two- and three-dimensions are given by $E=E_S(1+\beta^2/\alpha^2)$ and $E=E_S(1+\beta^3/2\alpha^3)$, respectively. Thus, more practical expressions can be devised in the forms:

$$\mathrm{Im}\left[G_{ij}(\mathbf{x},\mathbf{y},\omega)\right] \approx -\frac{(1+\beta^2/\alpha^2)}{4\mu}\left\langle \frac{u_i(\mathbf{y},\omega)}{\left|\mathbf{u}(\mathbf{y},\omega)\right|} \times \frac{u_j^*(\mathbf{x},\omega)}{\left|\mathbf{u}(\mathbf{x},\omega)\right|}\right\rangle, \text{ and} \tag{39}$$

$$\mathrm{Im}\left[G_{ij}(\mathbf{x},\mathbf{y},\omega)\right] \approx -\frac{(1+\beta^3/2\alpha^3)k}{2\pi\mu}\left\langle \frac{u_i(\mathbf{y},\omega)}{\left|\mathbf{u}(\mathbf{y},\omega)\right|} \times \frac{u_j^*(\mathbf{x},\omega)}{\left|\mathbf{u}(\mathbf{x},\omega)\right|} \right\rangle, \tag{40}$$

for two- and three-dimensions, respectively. The equipartition ratio, for Poisson coefficient of 0.25, is 3 and 10.4 in these two cases. Thus, the first factor in Eqs. 39 and 40 is of the order of one.

Equation 39 for 2D elasticity is formally applicable to the surface terms of Green functions in a layered space, at least for a singe mode. In Aki and Richards (1980) asymptotic expressions of the Green function are given in terms of normal modes for a unit force within a layered halfspace. Thus formal relationships exist between the horizontal components of Rayleigh and Love waves and their *in-plane* counterparts of P and SV waves, respectively. On the other hand, for the vertical displacements of Rayleigh waves the problem is analogous to the 2D scalar homogeneous problem (i=j=2). Because the symmetries are the same, it suffices a proper interpretation. The x and z axis should be horizontal, the y axis vertical and wavenumbers $q = \omega/c_R(\omega)$ and $k = \omega/c_L(\omega)$ would correspond to Rayleigh and Love waves. After tensor rotation, the longitudinal and transverse components will give the contributions of Rayleigh and Love waves. This analysis is similar of the one given by Snieder (2004). With the presence of various modes, equipartition will certainly arise in the diffusive regime but the precise relationship of correlation with Green function requires further scrutiny. In any case, the Fourier transform of correlations may reveal the various modes.

Lobkis and Weaver (2001) give a demonstration of the proportionality between Green function and correlation based on a modal approach that was later extended to open medium (Weaver and Lobkis, 2004). They considered a source distribution in their acoustic formulation. They did not require any sources in the neighborhood of receivers and extended the argument to apply to the far more useful case of the receiver region insonified by a conventionally diffuse equipartitioned set of random plane waves. Van Tiggelen (2003) gives a theoretical analysis of the reconstruction of the scalar Green function from correlation of the signals produced by a single source in a disordered medium with homogeneous background using the diffusion approximation. Wapenaar (2004) gives an interpretation based on a special kind of representation theorem and assumed a continuous distribution of sources on a closed surface surrounding the receivers. Using a heterogeneous acoustic model van Manem *et al*. (2005) actually tested this idea and compute the field with finite differences. Wapenaar's (2004) is a formal argument that is close to the one of the perfect time reversal mirror (Derode *et al*., 2003) but generalized to elasticity. The analogy with the time reversal, as well as Wapenaar's argument, indicates that the reconstructed signal, when it emerges correctly from the remnant fluctuations, is the actual Green function of the medium.

Snieder (2004) used a model for surface waves that assumes that multiple scattering is equivalent to the contribution of a random, uncorrelated (secondary) source distribution. Indeed, he obtained the surface waves in the correlation and gave a geometrical interpretation (stationary phase: constructive contributions from the 'sources' located in the direction defined by the two receivers). Snieder also concluded that Green function reconstruction has no relation with equipartition while we show here that in canonical cases, it is a clear requirement. The stationary phase argument is also strongly used by researchers from ocean acoustics since the early works by Kuperman and Ingenito (1980). For example, Roux *et al*. (2005) computed the correlation in a 3D acoustic medium with a homogeneous distribution of uncorrelated sources and get the 3D Green's function. Again dealing with a purely homogeneous background he gives a geometrical interpretation (stationary phase in a region around the direction of the receivers referred to as 'end fire lobes'). One must note that none of these last authors really deal with a Green function made of different contributions for which the relative amplitudes of the contributions are discussed. Our analytical results indicate that both equipartition and isotropy of the field are required to exactly retrieve the elastic Green function. Since equipartition is expected for multiply scattered waves, late coda records are good candidates for use in reconstruction of complete Green functions. Furthermore, the randomness of the field and particularly its isotropy can be improved by averaging the cross-correlation over a set of sources (see Paul *et al*., 2005 for a thorough discussion). Seismic noise on Earth has an origin that is not fully understood and there is no guarantee that noise has a homogeneous distribution of arrival directions or any kind of equipartition between the different waves since it can be dominated by ballistic arrivals. It can be isotropic enough to make it possible good reconstructions of specific arrivals, such as fundamental mode of surface waves. It was demonstrated that long range correlations in the noise can actually be used to retrieve the surface wave part of the Green functions (Shapiro and Campillo, 2004; Sabra *et al*. 2005) and even to produce tomographic maps (Shapiro *et al*. 2005). In spite of the practical success of these approaches, that shows the robustness of the method, its interpretation and the limits of the method are still open for discussion. It is noteworthy that the P and S energy densities equilibrate to the equipartition ratio before the field isotropy is reached (*e.g*. Paul *et al*., 2005). An anisotropic flux as well as the absence of equipartition has to be considered to fully understand the limitations of the method.

CONCLUSIONS

The equipartition of the energy carried by diffuse elastic waves in 3D leads to the relationship $E_S = 2(\alpha/\beta)^3 E_P$, where E_S and E_P are the S and P spatial energy densities, and α and β are the P and S wave speeds, respectively. In 2D that factor is simply $(\alpha/\beta)^2$. The case of the elastic, isotropic and homogeneous body under isotropic random distribution of plane waves is an important canonical problem. We retrieve from the field correlations the exact properties of Green function, like distance behavior of P and S waves and the precise balance between P and S energies. Even if there is not scattering at all, equipartition can be reached within a portion of a perfect elastic solid if random sources are isotropically distributed far away from the center of the station pair. The energy ratio between P and S in the diffuse incoming random field governs the balance in the correlation. In the elastic case, the particular value of this ratio that leads to the exact full Green function is precisely the one predicted by the theory of equipartition. Without equipartition we do not retrieve the exact Green function from the correlation. However, different arrivals can be usefully reconstructed with amplitude of each of them governed by the energy distribution among the modes of the recorded field.

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APPENDIX *Equipartition Ratio of Elastic Waves Within a Diffuse Field*

Diffuse field

The extension of some of the concepts of room acoustics to elastic waves lead to the consideration of diffuse fields (Eagle, 1981). One definition of a diffuse field at a given frequency establishes that at each point of the vibrating medium the disturbance be an isotropic random superposition of plane waves. Weaver (1982) extended such a definition of a diffuse field in a system and allowed it to be an excitation for which each normal mode with a natural frequency in the neighbourhood of that frequency is, statistically speaking, excited to equal energy. Additionally, there is no correlation in phase and amplitude between the degrees of excitation of different modes.

In what follows we explain the mode counting approach followed by Weaver (1982) to found the energy partition ratio between shear and dilatational waves in a diffuse field. Additionally, we present an alternative view regarding the issue from the perspective of plane waves. In both cases we give a self-contained account.

Weaver's Mode Counting Approach

The displacement field $u_i(\mathbf{x},t)$ within a homogeneous, isotropic, elastic medium must fulfil Navier's equation:

$$\mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + (\lambda + \mu)\frac{\partial^2 u_j}{\partial x_i \partial x_j} = \rho \frac{\partial^2 u_i}{\partial t^2}\ , \qquad \text{(A1)}$$

where, x_i stands for *x*, *y* and *z* (when *i*=1,2 and 3, respectively); λ, μ= Lamé constants, ρ=mass density, and *t*=time. Solutions of Eqn. A1 can be found by means of Helmholtz's decomposition:

$$u_i(\mathbf{x},t) = \frac{\partial \Phi}{\partial x_i} + \varepsilon_{ijk}\frac{\partial \Psi_k}{\partial x_j}, \text{ with } \frac{\partial \Psi_k}{\partial x_k} = 0, \tag{A2}$$

and Φ, Ψ_k are scalar and vector potentials (for P and S waves, respectively) and ε_{ijk} = permutation symbol (see *e.g.* Aki and Richards, 1980). The potential for P waves should fulfil the wave equation in three dimensions:

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = \frac{1}{\alpha^2}\frac{\partial^2 \Phi}{\partial t^2}, \tag{A3}$$

where $\alpha = \sqrt{(\lambda + 2\mu)/\rho}$ = P wave propagation velocity. Without loss of generality, let us assume standing P waves within a finite region (a cube with side L much larger than the wavelength) with Dirichlet boundary condition ($\Phi = 0$) at the edges so that we can give a modal solution of the form

$$\Phi = A \sin\frac{n_x \pi x}{L} \times \sin\frac{n_y \pi y}{L} \times \sin\frac{n_z \pi z}{L} \times \sin\omega t\,. \tag{A4}$$

Here $\omega = 2\pi\alpha/\Lambda$ = circular frequency, and Λ = wavelength of P waves. Substituting this solution into the wave equation (Eqn. A3) gives

$$\left[\frac{n_x \pi}{L}\right]^2 + \left[\frac{n_y \pi}{L}\right]^2 + \left[\frac{n_z \pi}{L}\right]^2 = \left[\frac{2\pi}{\Lambda}\right]^2, \tag{A5}$$

which can be simplified to

$$n_x^2 + n_y^2 + n_z^2 = \left[\frac{L\omega}{\pi\alpha}\right]^2. \tag{A6}$$

Our aim is to obtain the number of modes which can meet this condition. This amounts to counting all the possible combinations of the integer n values. This can be done approximately by treating the number of combinations as the volume of a sphere in the "n-space" multiplied by 1/8 to consider only the positive n's (see *e. g.* Kittel and Kroemer, 1980). Therefore, the number of dilatational modes is given by

$$N_d = \frac{1}{8} \times (\text{"Volume" of } n\text{'s}) = \frac{1}{8} \times \frac{4\pi}{3}(n_x^2 + n_x^2 + n_x^2)^{3/2} = \frac{\pi}{6}\left[\frac{L\omega}{\pi\alpha}\right]^3. \tag{A7}$$

This value becomes a very good approximation when the size of the control volume is much greater than the wavelength.

Regarding S waves, the arguments follow along the same lines using instead $\beta = \sqrt{\mu/\rho}$, the shear wave velocity. Additionally, we have waves polarized in two perpendicular planes, so we must multiply by two to account for that. We can write then the number of transverse modes as:

$$N_t = 2\times\frac{\pi}{6}(n_x^2+n_y^2+n_z^2)^{3/2} = \frac{\pi}{3}\left[\frac{L\omega}{\pi\beta}\right]^3 . \qquad \text{(A8)}$$

Having developed expressions for the numbers of dilatational and transverse waves in a large control volume $L^3=V$ it is of interest to know the distributions with frequency. This may be obtained by taking the derivative of the number of modes with respect to frequency:

$$\frac{dN_d}{d\omega} = (1/2\pi^2)\omega^2\alpha^{-3}V \text{ , and } \frac{dN_t}{d\omega} = 2(1/2\pi^2)\omega^2\beta^{-3}V, \qquad \text{(A9)}$$

for dilatational and transverse waves, respectively. These expressions assume no coupling between P and S waves. Weaver (1982) points out that this is not the case for the true normal modes of a finite solid. However, for a sufficiently large solid with $\omega L/c >> 1$ the coupling is weak. It tends to zero as V goes to infinity and thus the estimates become exact.

Let us make the diffuse-field assumption and assign energy to the elastic standing waves in a volume according to the principle of equipartition of energy. The energy associated to each state is then proportional to the number of modes. In this way, the ratio of the expected amounts of energy in transverse and dilatational waves in a small frequency interval $\Delta\omega$ is the ratio of N_t to N_d:

$$\frac{E_S}{E_P} = \frac{(dN_t/d\omega)\Delta\omega}{(dN_d/d\omega)\Delta\omega} = 2\left(\frac{\alpha}{\beta}\right)^3 . \qquad \text{(A10)}$$

This result is due to Weaver (1982).

In two-dimensions we deal with the mode counting in similar way. Instead of a “mode-volume” we have a “mode-area” and in-plane S waves have only one polarization.

Therefore, the number of dilatational and transverse modes in a given area $A=L^2$ are

$$N_d = \frac{1}{4}\times(\text{"Area" of } n\text{'s}) = \frac{1}{4}\times\pi(n_x^2+n_y^2) = \frac{\pi}{4}\left[\frac{L\omega}{\pi\alpha}\right]^2 \text{ and } N_t = \frac{\pi}{4}\left[\frac{L\omega}{\pi\beta}\right]^2 \text{, respectively.}$$

Then, using the same arguments above, the energy ratio in 2D becomes simply

$$\frac{E_S}{E_P} = \left(\frac{\alpha}{\beta}\right)^2 . \qquad \text{(A11)}$$

Up to here the equipartition concept *à la Weaver* is based upon the extension of the results from those of a finite body to and infinite domain.

Plane Waves Approach

If instead of modes we speak of plane waves defining the phase space, a simple argument can be given for the case of the homogeneous infinite space. In this case, the P and S plane waves form a continuous set of solutions of the Navier equation. Assuming equipartition in the phase space means an excitation of all plane waves at a constant energy level. At a given frequency ω, the phase space is defined by (q_x, q_y, q_z). The plane P waves are distributed on a sphere defined by

$$q_x^2 + q_x^2 + q_x^2 = (\omega / \alpha)^2, \tag{A12}$$

where α = P wavespeed. Similarly, S waves are distributed on a sphere defined by

$$k_x^2 + k_x^2 + k_x^2 = (\omega / \beta)^2, \tag{A13}$$

with two possible polarizations. Assuming a uniform distribution of energy, it comes out that for P waves in a narrow frequency band [ω, $\omega+\Delta\omega$] the energy is proportional to the volume of the thin shell containing the solutions:

$$E_P = 4\pi(\omega / \alpha)^2 \Delta q = 4\pi\omega^2\alpha^{-3}\Delta\omega, \tag{A14}$$

while for the two polarizations of S waves, the energy is:

$$E_S = 2 \times 4\pi(\omega / \beta)^2 \Delta k = 8\pi\omega^2\beta^{-3}\Delta\omega. \tag{A15}$$

The ratio of energy in 3D is therefore:

$$\frac{E_S}{E_P} = 2\left(\frac{\alpha}{\beta}\right)^3, \tag{A16}$$

and corresponds to Eq. A10. The argument for the 2D case follows in parallel.

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