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# Diffuse fields in dynamic elasticity

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## Abstract

In this article the problem of Green function retrieval from correlations is approached from a theoretical point of view and for this purpose an integral identity is considered: a representation theorem of the correlation type for an inhomogeneous region embedded in a homogeneous space. The full homogeneous case is studied with the theorem and it is concluded that, in the resulting field, the energy is equipartitioned. In infinite space this means that the ratio of P and S energy densities stabilizes to a constant value. That equipartition is reached in the classical sense is also demonstrated. Thus, in infinite space the energy densities associated with the possible degrees of freedom tend to share in equal parts the available energy.

The representation theorem permits the verification of the well known result that by averaging correlations of motions from diffuse, equipartitioned fields within an inhomogeneous, anisotropic, elastic medium it is possible to retrieve its Green function. As a result of this it is shown that the average autocorrelation of the diffuse displacement field at a point is proportional to the imaginary part of the Green function at the source precisely at this point. As a consequence, the energy density of the diffuse field is proportional to the trace of the imaginary part of the Green tensor at the source. Thus, the analytical form of the Green function permits the establishment, in and around an inhomogeneous region, of the theoretical energy density of a diffuse field.

In both homogeneous and inhomogeneous cases (*i.e.* localized elastic inclusions or cavities) the equipartition of the background illumination (the so called incident field in scattering theory) is a necessary and sufficient condition to retrieve the *exact* Green function from correlations. Local effects lead to energy ratios that fluctuate in space and frequency. The boundary of a half-space produces in its interior fluctuations of energy densities that are local effects of the diffuse field. These results may be useful to assess the diffuse nature of seismic ground motion from a limited set of observation points and to detect the presence of a target by its signature in the distribution of diffuse energy.


*Keywords:* Diffuse fields; Dynamic elasticity; Green function retrieval; Equipartition; Correlations; Noise; Energy densities

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## 1. Introduction

The pioneering work of Aki [1] has been crucial in the understanding of the role of coda waves and seismic noise. Aki considered single and multiple scattering formulations and explored Radiative Transfer concepts in order to explain coda envelopes [2]. The physics of multiple scattering is interesting. Although on the micro scale the field equations remain unchanged (*e.g.* in dynamic elasticity Newton and Hooke's laws lead to the governing equations), the intensities follow a diffusion equation. Evidence of the transition toward a diffusive regime has been observed. This appears in real data [3,4] in which the stabilization of the ratio of P- and S-energies in the coda has a value in agreement with equipartition theory. This happens with coda waves, which continue ringing for a duration which is much longer than the source–station travel time [5]. Because of multiple scattering, coda waves arrive at the station from different directions. We expect these waves to sample the medium more or less uniformly around the recording station. Therefore, coda waves qualify as a diffuse field and equipartition should arise naturally.

Equipartition means that in phase space the available energy is equally distributed, in fixed average amounts, among all the possible states. These ideas from thermodynamics have been introduced into acoustics and elastic wave propagation by Weaver [6]. In mono atomic gases the three degrees of freedom define the independent "states" and each one takes one third of the total energy. In diffuse dynamic elasticity we define "state" as an independent wave configuration. We can talk either of components (*i.e.* horizontal and vertical that each takes one half of the energy) or wave types (we have P and S waves to deal with and energy will be shared according to their respective wavelengths or wave speeds). In 2D we may have two components and, on the other side, the P and S waves represent all the possible configurations. In 3D there are three orthogonal directions (each with one third of the energy) and three wave types (the P and two possible polarizations for S waves: SH and SV). If only the degrees of freedom are considered, then equipartition is governed in infinite space by geometry. For example, in 2D or 3D the ratio $E_H/E_V = 1$ or 2, respectively, where $E_H$ and $E_V$ are the horizontal and vertical energy densities. In 3D this means that there are two independent horizontal directions.

Regarding wave types, the equipartition of energy in 2D or 3D for a homogeneous elastic medium leads to the relation $E_S/E_P = \alpha^2/\beta^2$ or $2\alpha^3/\beta^3$, respectively, where $E_S$ and $E_P$ are the S and P spatial energy densities and $\beta$, $\alpha$ are the respective wave speeds. This has been found by Weaver [6] by counting modes. Note that his approach is rigorous for closed systems, for which one can count the states and account for the energy in every state. For the open systems like the one we dealt with, there are no discrete states. This would mean that equipartitioning from mode counting is not strictly applicable. However, the relative number of states scales with the size ratio of the system but the energy ratio at the limit tends towards the theoretical value. The same result can be obtained using different arguments such as those of Snieder [7]. An interesting result was obtained using radiative transfer ideas: the energy ratio of the various wave types reaches a constant value, independent of the details of the scattering [8]. This stabilization of the elastic energy among the P and S waves has been verified by numerical simulations [25] and observations [3,4].

The Green function is the wave field that would be observed at one position if an impulsive load is applied at another [15]. This function, also called the fundamental solution, has been recovered experimentally from the averaging of the cross-correlation of the isotropic elastic wave field generated by either multiple scattering or by a large number of sources (such as microseisms). An isotropic elastic wave field can be understood as a uniform distribution of waves approaching the recording points from all possible directions. It also means that the incident field has equal average intensities in all directions. Approximate derivations, based on the concept of stationary phase, have been proposed (*e.g.* [13,30]). Further generalizations are due to Weaver [31].

Correlation of the seismic codas of 101 distant earthquakes recorded at different stations was done by Campillo and Paul [9]. In that study they made important comparisons between the recovered and theoretical Green function. Similar approaches for extracting information from the Earth's structure were followed by Shapiro and Campillo [10] and Sabra et al. [11]. They retrieved surface-wave dispersion curves from seismic noise measurements. A comprehensive discussion regarding the correlations and retrieval of the Green function in diffuse fields is due to Campillo [12]. He concluded that field–field correlation may lead to obtaining the deterministic response between two stations based on the actual wave propagation in the Earth.

The accuracy of the reconstructed Green function depends critically on the duration of the signals processed. The cross-correlations should be applied to equipartitioned fields that are in a diffusive regime in which the net energy flux is null. This takes place after enough time has elapsed to allow multiple scattering (and thus the diffusion) of the wave field. In recent experimental work with a ballistic ultrasound pulse launched in a highly heterogeneous rock sample, the emergence of the Green function was observed from correlations [14]. About nine mean-free times were required.

The canonical case of the homogeneous elastic medium both in 2D and 3D was recently considered [15]. In that study, isotropic illumination and equipartition of the propagating waves were assumed and it was demonstrated that the Fourier transform of the azimuthal average of the cross-correlation between the vector motions at two points within an infinite elastic space is proportional to the imaginary part of the exact Green tensor function between these points. These results exhibit equipartition of the field as necessary and sufficient to retrieve the *exact* Green tensor from correlations of the elastic field.

For general inhomogeneous media subjected to a diffuse field, Weaver and Lobkis [16] established a formal identity between the Green function and correlations of the diffuse field. The Green function that emerges from the correlations is found to be the complete Green function of the medium, symmetrized in time, having all reflections, scattered waves and the converted P and S waves as well.

In fact, in a recent study regarding a homogeneous medium with a 2D circular cylindrical elastic inclusion [17] the result for the homogeneous case [15] was extended and the theorem of Weaver and Lobkis [16] was verified. The cylinder was studied assuming uniform random distributions of plane waves coming from infinity. The cross-correlation at two points of the fields produced by the generic plane waves was computed, and then azimuthally averaged [17]. It was shown that the Fourier transform of the average of the cross-correlation of the vector motion between two points is proportional to the imaginary part of the Green tensor for these points. Again equipartition of the background field is a necessary and sufficient condition to retrieve the *exact* Green function from correlations [17]. As the cylindrical inclusion studied is embedded in a full space it is clear that the equipartitioned, isotropic illumination (a background radiation) is independent of the scatterer but the local equipartitioned, diffuse regime already includes its effects and fluctuates in space and frequency.

The study of inhomogeneous media and the half-space problem allows for fluctuations of the energy densities of a diffuse field. The partitioning of energy at the free surface was treated in detail by Weaver [18] who established explicit partition coefficients for energy densities. In this study we consider the 2D case for a half-space and a cylindrical cavity as well.

In what follows a correlation type representation theorem for an inhomogeneous region embedded in a homogeneous space is considered. Assuming homogeneity this integral identity simplifies. It is found that for uncorrelated boundary sources the resulting field is equipartitioned.

The average autocorrelation of the diffuse displacement field is shown to be proportional to the imaginary part of the Green function at the source. Therefore, the trace of the imaginary part of the Green tensor at the source is proportional to the kinetic energy density of the field. Thus, the analytical form of the Green function may allow the establishment of the theoretical energy density of a diffuse field generated by background equipartitioned illumination. These results may be useful in assessing the diffuse nature of seismic ground motion from a limited set of observation points and in detecting the presence of a target by its signature in the distribution of diffuse energy.

## 2. Representation theorem of the correlation type

Consider the harmonic displacement field $u_i(\mathbf{x},\omega)$ produced by a harmonic body force distribution $f_j(\mathbf{x},\omega)$ within an arbitrary heterogeneous elastic medium. The displacements satisfy the elastic wave equation:

$$\frac{\partial}{\partial x_j}\left(c_{ijkl}\frac{\partial u_l(\mathbf{x},\omega)}{\partial x_k}\right)+\omega^2\rho u_i(\mathbf{x},\omega)=-f_i(\mathbf{x},\omega), \tag{1}$$

where $c_{ijkl}$ is the elastic tensor that may vary over the space, $\rho$ is the mass density and $\omega$ is the angular frequency. As usual, the summation convention is used (a repeated index, twice and only twice, means summation over the range of such index).

Consider the case of a concentrated harmonic unit force at $\mathbf{x_A}$ in the $m$ direction. This force can be written as $f_i(\mathbf{x},\omega) \equiv \delta(\mathbf{x}-\mathbf{x_A})\delta_{im}\exp(\mathrm{i}\omega t)$. The index $m$ is added to specify the direction of the concentrated force and to define the resulting displacements as the Green function. Thus, Eq. (1) becomes:

$$\frac{\partial}{\partial x_j}\left(c_{ijkl}\frac{\partial G_{lm}(\mathbf{x},\mathbf{x_A},\omega)}{\partial x_k}\right)+\omega^2\rho G_{im}(\mathbf{x},\mathbf{x_A},\omega)=-\delta(\mathbf{x}-\mathbf{x_A})\delta_{im}. \tag{2}$$

The Green function $G_{im}(\mathbf{x},\mathbf{x_A},\omega)$ is the displacement at $\mathbf{x}$ in direction $i$ produced by a unit harmonic point force acting at $\mathbf{x_A}$ in direction $m$.

Multiplying Eq. (1) with $G_{im}(\mathbf{x},\mathbf{x_A},\omega)$ and Eq. (2) with $u_i(\mathbf{x},\omega)$, and subtracting we have

$$\frac{\partial}{\partial x_j}\left(c_{ijkl}\frac{\partial G_{lm}(\mathbf{x},\mathbf{x_A})}{\partial x_k}\right)u_i(\mathbf{x},\omega)-\frac{\partial}{\partial x_j}\left(c_{ijkl}\frac{\partial u_l(\mathbf{x})}{\partial x_k}\right)G_{im}(\mathbf{x},\mathbf{x_A})=-\delta(|\mathbf{x}-\mathbf{x_A}|)u_m(\mathbf{x})+f_i(\mathbf{x})G_{im}(\mathbf{x},\mathbf{x_A}). \tag{3}$$

The dependence on $\omega$ is understood and is omitted henceforth. Integrating over a volume $V$ bounded by a surface $\Gamma$, and using the Gauss formula and reciprocity, it follows that:

$$\int_\Gamma\left\{n_j(\mathbf{x})\left(c_{ijkl}\frac{\partial G_{lm}(\mathbf{x},\mathbf{x_A})}{\partial x_k}\right)u_i(\mathbf{x})-n_j(\mathbf{x})\left(c_{ijkl}\frac{\partial u_l(\mathbf{x})}{\partial x_k}\right)G_{im}(\mathbf{x},\mathbf{x_A})\right\}\mathrm{d}\Gamma_\mathbf{x}$$
$$=-u_m(\mathbf{x_A})+\int_V f_i(\mathbf{x})G_{im}(\mathbf{x},\mathbf{x_A})\,\mathrm{d}V_\mathbf{x}. \tag{4}$$

Note that $\mathbf{x_A}$ is a point inside $V$. We recognize

$$t_i(\mathbf{x})=n_j(\mathbf{x})\left(c_{ijkl}\frac{\partial u_l(\mathbf{x})}{\partial x_k}\right), \tag{5}$$

as the traction at point $\mathbf{x}$, with normal $n_j(\mathbf{x})$, in direction $i$, associated with displacement field $u_l(\mathbf{x})$ while

$$T_{im}(\mathbf{x},\mathbf{x_A})=n_j(\mathbf{x})\left(c_{ijkl}\frac{\partial G_{lm}(\mathbf{x},\mathbf{x_A})}{\partial x_k}\right), \tag{6}$$

is the traction at point $\mathbf{x}$, with normal $n_j(\mathbf{x})$, in direction $i$ produced by the unit harmonic point force acting at $\mathbf{x_A}$ in direction $m$ associated with the Green function.

After some rearrangement, Eq. (4) can then be written as

$$u_m(\mathbf{x_A})=\int_\Gamma\{G_{im}(\mathbf{x},\mathbf{x_A})t_i(\mathbf{x})-T_{im}(\mathbf{x},\mathbf{x_A})u_i(\mathbf{x})\}\,\mathrm{d}\Gamma_\mathbf{x}+\int_V f_i(\mathbf{x})G_{mi}(\mathbf{x_A},\mathbf{x})\,\mathrm{d}V_\mathbf{x}. \tag{7}$$

This is the classical Betti–Rayleigh reciprocity identity. It is also known as the Somigliana's representation theorem. It expresses the relationship between the boundary values of displacements and tractions and internal displacements.

Now consider for excitation a harmonic body force at the internal point $\mathbf{x_B}$ in the direction $n$ and assume that both displacements and tractions are time-reversed. Eq. (7) is symmetrical in time and admits time-reversed solutions. Time-reversal leads to complex conjugate values in the frequency domain. One can then write

$$f_i(\mathbf{x})\equiv\delta(\mathbf{x}-\mathbf{x_B})\delta_{in}, \tag{8}$$

with

$$u_i(\mathbf{x})\equiv G^*_{in}(\mathbf{x},\mathbf{x_B})\quad\text{and}\quad t_i(\mathbf{x})\equiv T^*_{in}(\mathbf{x},\mathbf{x_B}). \tag{9}$$

Substituting Eqs. (8) and (9) in Eq. (7), we can write

$$\int_\Gamma\left\{T_{im}(\mathbf{x},\mathbf{x_A})G^*_{in}(\mathbf{x},\mathbf{x_B})-T^*_{in}(\mathbf{x},\mathbf{x_B})G_{im}(\mathbf{x},\mathbf{x_A})\right\}\mathrm{d}\Gamma_\mathbf{x}=-G^*_{mn}(\mathbf{x_A},\mathbf{x_B})+G_{mn}(\mathbf{x_A},\mathbf{x_B}), \tag{10}$$

which is re-written changing $\mathbf{x}$ by $\xi$, to represent boundary points on $\Gamma$, by means of

$$2\mathrm{i}\,\mathrm{Im}[G_{mn}(\mathbf{x_A},\mathbf{x_B})] = -\int_{\Gamma}\left\{G_{mi}(\mathbf{x_A},\xi)T^{*}_{in}(\xi,\mathbf{x_B}) - G^{*}_{ni}(\mathbf{x_B},\xi)T_{im}(\xi,\mathbf{x_A})\right\}\mathrm{d}\Gamma_{\xi}. \tag{11}$$

This expression was presented by Wapenaar [19] and van Manen et al. [20] with a somewhat different notation. The result is implicit in the treatment by Weaver and Lobkis [16]. This identity expresses the imaginary part of the Green tensor between points $\mathbf{x_A}$ and $\mathbf{x_B}$. In the time domain it gives the Green tensor plus the time-reversed negative mirror image. Eq. (11) is thus a perfect *time-reversal device* for an arbitrary inhomogeneous medium within the boundary $\Gamma$ (see [20,22]). The concept has been used extensively by Fink [21] and others (*e.g.* [22,23]) regarding a *time-reversal mirror* in which the time symmetry of wave equation is exploited. On the other hand, the inverse transformation of the imaginary part of the Fourier transform of a causal function leads to one half the function minus one half the time-reversal counterpart.

It is worth mentioning that time-reversal, which is frequently invoked in Green function retrieval, is not fulfilled by the diffusion equation. However, Snieder [24] showed recently that the Green function of the diffusion equation can be retrieved from the response to random forcing.

## 3. The isotropic illumination or the incoming field

To introduce some concepts, consider first the special case of an inhomogeneous medium, which may be as extensive as necessary, surrounded by a homogeneous space and that both $\mathbf{x_A}$ and $\mathbf{x_B}$ are far away from the inhomogeneous region and from the boundary $\Gamma$ as well (see Fig. 1). A similar approach was taken by Weaver and Lobkis [16] and it is equivalent to considering only the homogeneous problem and assuming the elastic field under scrutiny as a background field. Expressions for the homogeneous case that will share the structure of the inhomogeneous problem are presented.

From the exact expressions in 2D and 3D (see *e.g.* [15]), the far field asymptotic expressions of the Green tensors, for both displacements and tractions in a homogeneous, isotropic, elastic medium are given by

$$G_{mi}(\mathbf{x_A},\xi) \approx f_1(qr_{\mathbf{A}})n_m n_i + f_2(kr_{\mathbf{A}})(\delta_{mi} - n_m n_i) \text{ and} \tag{12}$$

$$T_{in}(\xi,\mathbf{x_B}) \approx \mathrm{i}\omega\rho[\alpha f_1(qr_{\mathbf{B}})n_i n_n + \beta f_2(kr_{\mathbf{B}})(\delta_{in} - n_i n_n)], \tag{13}$$

respectively. In these equations $r_{\mathbf{A}}$ and $r_{\mathbf{B}}$ are the distances between the point $\xi$ at the boundary $\Gamma$ and points $\mathbf{x_A}$ and $\mathbf{x_B}$. As the distances are very large, we assumed the unit vector $n_j$ at the surface $\Gamma$ to be equal for both points. These expressions have an interesting form: the product $f_1(qr_{\mathbf{A}})n_m n_i$ is a tensor that represents the longitudinal part of the motion while $f_2(kr_{\mathbf{A}})(\delta_{mi} - n_m n_i)$ accounts for the transverse part. Notice that in Eq. (13) the tractions at $\xi$ are formed by the sum of the paraxial radiation boundary conditions for P and S waves, respectively.

The radial functions $f_1$ and $f_2$ in 2D are cylindrical functions and are given by

$$f_1(qr_{\mathbf{A}}) = \frac{1}{4\mathrm{i}\rho\alpha^2}H_0^{(2)}(qr_{\mathbf{A}}) \quad \text{and} \quad f_2(kr_{\mathbf{A}}) = \frac{1}{4\mathrm{i}\rho\beta^2}H_0^{(2)}(kr_{\mathbf{A}}), \tag{14}$$

while in 3D these functions have spherical decay and it can be seen that

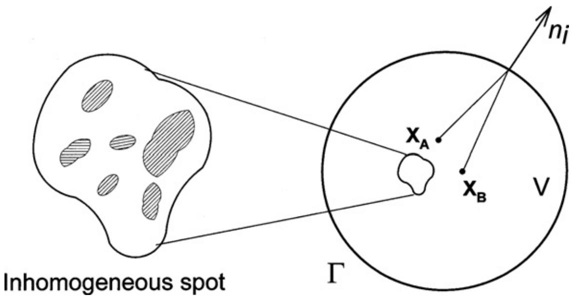


Fig. 1. Inhomogeneous spot within a homogeneous body.

$$f_1(qr_{\mathbf{A}}) = \frac{1}{4\pi\rho\alpha^2 r_{\mathbf{A}}}\exp(-\mathrm{i}qr_{\mathbf{A}}) \quad \text{and} \quad f_2(kr_{\mathbf{A}}) = \frac{1}{4\pi\rho\beta^2 r_{\mathbf{A}}}\exp(-\mathrm{i}kr_{\mathbf{A}}). \tag{15}$$

Here $H_0^{(2)}(\cdot)$ is the Hankel function of zero order and second kind, $q = \omega/\alpha$ and $k = \omega/\beta$, the wavenumbers, $\omega$ is the angular frequency, $\alpha$ and $\beta$ are the P and S wave speeds, respectively, $\rho$ is the mass density and $r_{\mathbf{A}}$ is the distance between $\xi$ and $\mathbf{x_A}$.

Consider the neighborhood of points $\mathbf{x_A}$ and $\mathbf{x_B}$ (Fig. 2). Let $r = |\ \mathbf{x_A} - \mathbf{x_B}\ |$ and $\gamma_j$ the unit vector in the direction from $\mathbf{x_A}$ to $\mathbf{x_B}$. Since $f_1$ and $f_2$ correspond to P and S waves, we can express $f_1(qr_{\mathbf{B}})$ and $f_2(kr_{\mathbf{B}})$ in the far field approximation in the form of plane waves as

$$f_1(qr_{\mathbf{B}}) = f_1(qr_{\mathbf{A}})\exp(-\mathrm{i}qr\gamma_j n_j) \quad \text{and} \quad f_2(kr_{\mathbf{B}}) = f_2(kr_{\mathbf{A}})\exp(-\mathrm{i}kr\gamma_j n_j) \tag{16}$$

in both 2D and 3D.

Substituting $f_1$ and $f_2$ of Eq. (14) or (15) in Eq. (11), and considering Eq. (16), we obtain

$$2\mathrm{i}\,\mathrm{Im}[G_{mn}(\mathbf{x_A},\mathbf{x_B})] = -2\mathrm{i}\omega\rho\int_\Gamma \left\{\alpha f_1^{\mathbf{A}} f_1^{\mathbf{B}*} n_m n_n + \beta f_2^{\mathbf{A}} f_2^{\mathbf{B}*}(\delta_{mn} - n_m n_n)\right\} \mathrm{d}\Gamma_\xi, \tag{17}$$

where $f_1^{\mathbf{A}} = f_1(qr_{\mathbf{A}})$ and $f_2^{\mathbf{A}} = f_2(kr_{\mathbf{A}})$.

The asymptotic forms of Hankel's functions for large arguments and Eq. (16) allow writing $f_1^{\mathbf{A}} f_1^{\mathbf{B}*} \approx (2/\pi)(qr_{\mathbf{A}})^{-1}\exp(\mathrm{i}qr\gamma_j n_j)$ and $f_2^{\mathbf{A}} f_2^{\mathbf{B}*} \approx (2/\pi)(kr_{\mathbf{A}})^{-1}\exp(\ \mathrm{i}kr\gamma_j n_j)$. The surface element can be written $\mathrm{d}\Gamma_\xi = r_{\mathbf{A}}\mathrm{d}\theta = 2\pi r_{\mathbf{A}}\mathrm{d}\theta/2\pi$. After some simplification, Eq. (17) can be expressed as

$$\mathrm{Im}[G_{mn}(\mathbf{x_A},\mathbf{x_B})] = -\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{4\rho}\left\{\frac{\exp(\mathrm{i}qr\gamma_j n_j)}{\alpha^2} n_m n_n + \frac{\exp(\mathrm{i}kr\gamma_j n_j)}{\beta^2}(\delta_{mn} - n_m n_n)\right\}\mathrm{d}\theta. \tag{18}$$

This angular integral has been evaluated before [15]. It is the azimuthal average of the cross-correlation of displacements $u_m(\mathbf{x_A})$ and $u_n(\mathbf{x_B})$ which can be represented using brackets: $\langle u_m(\mathbf{x_A})u_n^*(\mathbf{x_B})\rangle$. If this average is produced by the uniform isotropic illumination of plane P and S waves in such a way that their corresponding spectral densities satisfy $P^2\alpha^2 = S^2\beta^2 = 2$, we have the exact result

$$\mathrm{Im}[G_{mn}(\mathbf{x_A},\mathbf{x_B})] = \frac{-1}{8\rho}\{A\delta_{mn} - B(2\gamma_m\gamma_n - \delta_{mn})\}, \tag{19}$$

where $A = \frac{J_0(qr)}{\alpha^2} + \frac{J_0(kr)}{\beta^2}$ and $B = \frac{J_2(qr)}{\alpha^2} - \frac{J_2(kr)}{\beta^2}$ are given in terms of Bessel functions of zeroth and second order, respectively [15]. Therefore, the azimuthal averages of the cross-correlations of vector elastic motions of a diffuse field can formally be written as

$$\langle u_m(\mathbf{x_A})u_n^*(\mathbf{x_B})\rangle = -8\rho\frac{S^2\beta^2}{2}\times\mathrm{Im}[G_{mn}(\mathbf{x_A},\mathbf{x_B})], \tag{20}$$

where the brackets mean azimuthal average and $S^2$ the average spectral density of the plane shear waves of the isotropic background field. It is from this relationship that one can talk of Green function retrieval from cross-correlations [9–12]. In the canonical elastic case [15] the energies for S and P waves were assumed to fulfill the ratio $\alpha^2/\beta^2$, which is the theoretical equipartition ratio in 2D. In the sequel, instead, this ratio comes out nat-

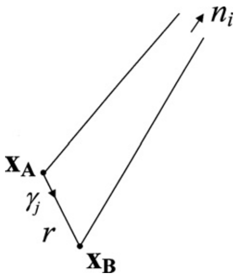


Fig. 2. Neighborhood of points $\mathbf{x_A}$ and $\mathbf{x_B}$.

urally from the representation theorem of Eq. (11). For completeness, we give here the result for the simple anti-plane SH case: $\text{Im}[G_{22}(\mathbf{x_A},\mathbf{x_B})] = -J_0(kr)/4\mu$.

From the results above we see that both $\mathbf{x_A}$ and $\mathbf{x_B}$ are arbitrary internal points. They can even be the same point. In that case, the imaginary part of the Green function is not singular. The singularity of the Green function at the origin corresponds only to the real part. Indeed, at $\mathbf{x_A} = \mathbf{x_B}$, $r = 0$ and we have, either from Eqs. (18) or (19), the expression

$$\text{Im}[G_{mn}(\mathbf{x_A},\mathbf{x_A})] = \frac{-1}{8\rho}\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}\right)\delta_{mn}, \tag{21}$$

which exhibits the imaginary part of the Green tensor *at the source* as a constant isotropic second order tensor. This means that all directions have the same value. The imaginary part of the Green function can be seen as directly proportional to the power injected into the medium by the unit harmonic load. Moreover, the parenthesis in Eq. (21) breaks up into two terms that strongly suggest the theoretical partition of the elastic energy density deep in infinite space. We assume this is the case.

Let us consider the identity of Eq. (20) that has been established between the Fourier transform of the azimuthal average of the cross-correlation of the vector motion at two points and the imaginary part of the Green tensor between these points [15]. Making coincident source and receiver, contracting the tensor (*i.e.* taking the two indexes to be the same which means summation over the said index. In 2D we have $\delta_{jj} = \delta_{11} + \delta_{33} = 2$) and multiplying with $\frac{1}{2}\rho\omega^2$ makes clear that the imaginary part of the trace of the Green tensor is proportional to the kinetic energy density. However, as the total energy is twice the average kinetic energy, we omit here and hereafter explicit reference to kinetic energy and write

$$\begin{aligned} E &= \rho\omega^2\langle u_m(\mathbf{x_A})u_m^*(\mathbf{x_A})\rangle \\ &= -4\mu E_S \times \text{Im}[G_{mm}(\mathbf{x_A},\mathbf{x_A})] \\ &= E_S \times \left(\frac{\beta^2}{\alpha^2}+1\right) = E_P + E_S, \end{aligned} \tag{22}$$

where $E$ is the total energy density, $E_S = \rho\omega^2 S^2$ and $E_P = \rho\omega^2 P^2$ are the average energy densities of the transverse and longitudinal waves, respectively. Clearly $E_S/E_P = \alpha^2/\beta^2$ and this is the value predicted by equipartition theory in 2D (see *e.g.* [6,15]).

This result takes into account only the wave type and the energy partition depends essentially upon wave length and the dimensions of space. Given the same (large) area $L^2$ the number of modes of P waves is approximately $N_P = \frac{\pi}{4}(\frac{\omega L}{\pi\alpha})^2$, which is 1/4 of the "area" of a circle with radius $\omega L/\pi\alpha$ in the modal space; see [6]. Thus, the density of P modes per unit surface at a narrow frequency band $\Delta\omega$ centered on the frequency $\omega$ is simply $(1/2\pi)\omega\ \Delta\omega/\alpha^2$ and, within the diffuse regime, it is proportional to the energy associated with P waves. The density of S modes is, instead, dependent on $1/\beta^2$ and therefore $E_S/E_P = \alpha^2/\beta^2$.

Equipartition theory also applies to the degrees of freedom and thus, when the isotropy of the tensor $\delta_{mn}$ of Eq. (21) is considered, it is clear that for any two directions $m$ and $n$ in infinite space the associated average energies will be the equal and thus $E_m/E_n = 1$. In other words, the energy associated with any given direction (in 2D) will be one half of the available energy density.

In three dimensions the concepts presented here permit expressing Eq. (11) by means of

$$\text{Im}[G_{mn}(\mathbf{x_A},\mathbf{x_B})] = \frac{-\omega}{16\pi^2 r_{\mathbf{A}}^2\rho}\int_\Gamma\left\{\frac{\exp(iqr\gamma_j n_j)}{\alpha^3}n_m n_n + \frac{\exp(ikr\gamma_j n_j)}{\beta^3}(\delta_{mn}-n_m n_n)\right\}d\Gamma_\xi, \tag{23}$$

where $d\Gamma_\xi = r_{\mathbf{A}}^2\, d\varphi\, \sin\theta\, d\theta$. This equation is in fact an identity [15]. Doing the tensor contraction (*e.g.* $n_j n_j = 1$ and $\delta_{jj} = 3$) and considering $r = 0$ we have

$$E = A \times \text{Im}[G_{nn}(\mathbf{x_A},\mathbf{x_A})] = A \times \frac{-\omega}{4\pi\rho}\left(\frac{1}{\alpha^3}+\frac{2}{\beta^3}\right) = E_P + E_S, \tag{24}$$

where $A = -2\pi\mu E_S k^{-1}$. It is clear that in 3D the energy ratio is $E_S/E_P = 2\alpha^3/\beta^3$. Again, this is agreement with equipartition theory [6,15]. Eq. (24) is implicit in Weaver's result [18] (see Eq. (62) in that paper). Regarding the degrees of freedom, it is clear that for any three directions $m$ and $n$ in infinite space the average energies will be equal and thus the energy associated with any given direction will be one third of the total energy density.

This section ends with a general comment and some considerations of energy partitions. The application of the correlation representation theorem of Eq. (11) for a homogeneous, isotropic, elastic medium shows that the average correlation of the motion in a diffuse regime is proportional to the imaginary part of the Green function. This is a consequence of the isotropic illumination of plane waves in such a way that the energy associated with P and S waves is equipartitioned.

In a homogeneous infinite space the average energy density $E$ can be distributed in different ways although the energy budget is fixed. To establish those quantities let us follow Weaver [18] who introduced partition coefficients. As kinetic and potential energies take one half each we can write $\mathrm{KE} = E \times G_{\mathrm{Kin}}$ and $\mathrm{PE} = E \times G_{\mathrm{Pot}}$, where the partition factors are $G_{\mathrm{Kin}} = G_{\mathrm{Pot}} = 1/2$. On the other hand, if the partition is made taking into account the degrees of freedom one has $E_i = E \times G_i$ and $G_i = 1/2$ or $1/3$ in 2D or 3D, respectively.

Up to now the different types of waves have been considered and have that $E_{\mathrm{P}} = E \times G_{\mathrm{P}}$ and $E_{\mathrm{S}} = E \times G_{\mathrm{S}}$ where $G_{\mathrm{P}} = (1+R^2)^{-1}$or$(1+2R^3)^{-1}$ in 2D or 3D, respectively. Here $R = \alpha/\beta$. Regarding the partition factor for S waves we have $G_{\mathrm{S}} = R^2(1+R^2)^{-1}$ or $2R^3(1+2R^3)^{-1}$ in 2D or 3D, respectively. Note that in 3D we have to account for the two possible polarizations: $G_{\mathrm{SH}} = G_{\mathrm{SV}} = R^3(1+2R^3)^{-1}$.

## 4. An inhomogeneous medium embedded in a homogeneous space

As the theorem of Eq. (11) is valid for any surface $\Gamma$, it follows that if the field is diffuse at the envelope, it will also be diffuse at any point within the heterogeneous medium. We are accepting that diffusion implies that the net flux of energy is null. Actually, it must be noticed that the stabilization of the P to S energy ratio, and the validity of the diffusion approximation occur for finite lapse times while perfect equipartition, associated with isotropy in our case, is reached asymptotically for long lapse times [25].

Consider an elastic inhomogeneous spot of arbitrary shape within a homogeneous envelope. The tractions at point $\xi$ on $\Gamma$ for a unit point load in the direction $n$ applied at point $\mathbf{x_B}$ can be written using the paraxial boundary conditions for both longitudinal and transverse components as

$$\begin{aligned} T_{in}(\xi, \mathbf{x_B}) &\approx -\mathrm{i}\omega\rho\big(\alpha G_{jn}(\xi, \mathbf{x_B}) n_j n_i + \beta G_{jn}(\xi, \mathbf{x_B})(\delta_{ji} - n_j n_i)\big) \\ &= -\mathrm{i}\omega\rho(\alpha n_j n_i + \beta(\delta_{ji} - n_j n_i)) \times G_{nj}(\mathbf{x_B}, \xi). \end{aligned} \tag{25}$$

Therefore, Eq. (11) can be written as

$$\mathrm{Im}[G_{mn}(\mathbf{x_A}, \mathbf{x_B})] = -\omega\rho \int_\Gamma (\alpha n_i n_j + \beta(\delta_{ij} - n_i n_j)) \times G_{mi}(\mathbf{x_A}, \xi) G^*_{nj}(\mathbf{x_B}, \xi)\, \mathrm{d}\Gamma_\xi. \tag{26}$$

Because the inhomogeneous region is far away from the boundary $\Gamma$, the resulting Green tensors, at that boundary, share the asymptotic properties of functions $f_1$ and $f_2$ as defined in Eqs. (12)–(15).

On the other hand, the elastic field at a point within $V$ can be expressed using a single-layer representation (this term comes from potential theory and means that the field can be constructed only by Green functions radiating from the boundary as opposed to the double-layer in which the field is composed of Green function derivatives. The first term of Somigliana's identity in Eq. (7) shows the elastic field to be the sum of single- and double-layer representations. See [26,27] for a discussion and applications to 2D and 3D problems). The field within $V$ can be represented by the elastic radiation of a force density $\phi_i(\xi)$ acting along $\Gamma$:

$$u_m(\mathbf{x}) = \int_\Gamma G_{mi}(\mathbf{x}, \xi)\phi_i(\xi)\, \mathrm{d}\Gamma_\xi. \tag{27}$$

It is usual to define a diffuse field (see [16,19]) in terms of a given force density such that its average along $\Gamma$ is null; here we represent the average with brackets, thus $\langle \phi_i(\xi) \rangle = 0$, and assume that $\phi_i(\xi)$ and $\phi_j(\xi)$ are mutually uncorrelated. Then we have

$$\langle \phi_i(\xi)\phi^*_j(\varsigma) \rangle = F^2 \delta_{ij} \delta(\xi - \varsigma), \tag{28}$$

where $F^2$ is the spectral density of the excitation.

The averaged cross-correlation of motion at points $\mathbf{x_A}$ and $\mathbf{x_B}$ is then given by

$$\langle u_m(\mathbf{x_A})u_n^*(\mathbf{x_B})\rangle = \int\int_\Gamma G_{mi}(\mathbf{x_A},\xi)G_{nj}^*(\mathbf{x_B},\varsigma)\langle\phi_i(\xi)\phi_j^*(\varsigma)\rangle\,\mathrm{d}\Gamma_\xi\,\mathrm{d}\Gamma_\varsigma. \tag{29}$$

Therefore,

$$\langle u_m(\mathbf{x_A})u_n^*(\mathbf{x_B})\rangle = F^2\int_\Gamma G_{mi}(\mathbf{x_A},\xi)G_{ni}^*(\mathbf{x_B},\xi)\,\mathrm{d}\Gamma_\xi. \tag{30}$$

From Eqs. (26) and (30), taking into account Eqs. (12)–(15) it is possible to write

$$\langle u_i(\mathbf{x_A},\omega)u_j^*(\mathbf{x_B},\omega)\rangle = -4E_\mathrm{S}k^{-2}\,\mathrm{Im}[G_{ij}(\mathbf{x_A},\mathbf{x_B},\omega)] \text{ and} \tag{31}$$

$$\langle u_i(\mathbf{x_A},\omega)u_j^*(\mathbf{x_B},\omega)\rangle = -2\pi E_\mathrm{S}k^{-3}\,\mathrm{Im}[G_{ij}(\mathbf{x_A},\mathbf{x_B},\omega)], \tag{32}$$

for 2D and 3D, respectively. Eqs. (31) and (32) are analytical consequences of the representation theorem (11) and have been verified recently for the full space [15] and an elastic inclusion embedded in an elastic space [17].

## 5. Fluctuations of energy densities near boundaries

In what follows we explore the energy density fluctuations at or near boundaries and scatterers. To this end let us rewrite Eqs. (31) and (32) as in Eq. (22):

$$E(\mathbf{x_A}) = \rho\omega^2\langle u_m(\mathbf{x_A})u_m^*(\mathbf{x_A})\rangle = -4\mu E_\mathrm{S}\times\mathrm{Im}[G_{mm}(\mathbf{x_A},\mathbf{x_A})] \text{ and} \tag{33}$$

$$E(\mathbf{x_A}) = \rho\omega^2\langle u_m(\mathbf{x_A})u_m^*(\mathbf{x_A})\rangle = -2\pi\mu E_\mathrm{S}k^{-1}\times\mathrm{Im}[G_{mm}(\mathbf{x_A},\mathbf{x_A})], \tag{34}$$

in 2D and 3D, respectively. In these equations $E_\mathrm{S} = \rho\,\omega^2 S^2$ = the energy density of shear waves which is a measure of the strength of the diffuse illumination. We see that the total energy density at a point is proportional to the imaginary part of the trace of the Green tensor for coincident receiver and source. As we will use $E(\mathbf{x})$ for the energy density at a point $\mathbf{x}$, let us call $E_\infty$ the energy density for the infinite space. It is clear that $E_\infty = E_\mathrm{P} + E_\mathrm{S}$ for elastic waves. A comment on the units used is necessary because Eqs. (33) and (34) are concerned with energies; the product of the shear modulus and Green function in 2D is non dimensional while in 3D the factor $k^{-1}$ (length) gives the dimensional homogeneity as well.

The free surface problem for a diffusively vibrating elastic body was studied by Weaver [18]. He defined partition coefficients for the energy densities of surface motions due to P, SH, SV and Rayleigh waves, respectively. In fact, Weaver [18] showed that a diffuse field at the free surface may be regarded as a summation of incoherent, isotropic and homogeneous independent plane waves incident upon the surface, together with their respective outgoing reflected consequences including the Rayleigh surface waves as well. Weaver's coefficients were evaluated for 3D at the free surface against Poisson ratios and have no frequency dependence.

In the next section the 2D half-space and a cylindrical cavity are considered. The fluctuations of energy densities near the free boundary for incident P, SH, SV and Rayleigh waves are analyzed.

### *5.1. The 2D half-space: anti-plane SH problem*

Consider first the anti-plane SH case and a half-space in 2D. The Green function can be obtained easily by superimposing the mirror image of the reflection. Thus, we can write

$$G_{22}(\mathbf{x_A},\mathbf{x_B}) = \frac{1}{4\mathrm{i}\mu}\left\{H_0^{(2)}(kr) + H_0^{(2)}(kr')\right\}, \tag{35}$$

where $H_0^{(2)}(\cdot)$ = cylindrical Hankel function of order zero and the second kind = $J_0(\cdot) - \mathrm{i}\,Y_0(\cdot)$, here $J_0(\cdot)$ and $Y_0(\cdot)$ = Bessel functions of order zero of the first and second kinds. Therefore, the imaginary part of the Green function is given by

$$\mathrm{Im}[G_{22}(\mathbf{x_A},\mathbf{x_B})] = \frac{-1}{4\mu}(J_0(kr) + J_0(kr')), \tag{36}$$

where $r$ is the distance between source and receiver and $r'$ is the distance between the image source and receiver. At the source $r = 0$ and $r' = 2z$. According to Eq. (33) the energy density is proportional to the imaginary part of the Green function at the source point. As we have only SH waves we have $E_\infty = E_S$. Therefore, recalling that $J_0(0) = 1$, one can write

$$E(z,\omega) = E_\infty \times (1 + J_0(2kz)). \tag{37}$$

This expression gives the energy density as a function of both frequency and distance to the free surface. At the surface the energy density is constant and twice the value of the infinite space. In Fig. 3, we depict the ratio $E(kz)/E_\infty$ against $kz$, a normalized depth because $k = 2\pi/\Lambda$ with $\Lambda =$ wavelength of shear waves.

This result shows that for a SH diffuse field in 2D the energy density at the surface is twice the reference level ($E_\infty$). In any event, the ratio depicted in Fig. 3 makes clear frequency dependent fluctuations with depth.

### 5.2. The 2D half-space: in-plane P, SV and Rayleigh waves

Consider the 2D elastic half-space but now with in-plane loading. Expressing the Green function in terms of standard horizontal wavenumber integrals (see [28]) it is possible to show for the source at depth $z_0$ that

$$\begin{aligned}\mathrm{Im}[G_{11}(\mathbf{x},\mathbf{x}) + G_{33}(\mathbf{x},\mathbf{x})] = {} & \frac{-1}{4\pi\rho\omega^2}\int_{-\infty}^{\infty}\mathrm{Re}(-\nu - k^2/\gamma - k\mathrm{e}^{-\mathrm{i}\gamma z_0}A_2 + \nu\mathrm{e}^{-\mathrm{i}\nu z_0}B_2)\,\mathrm{d}k \\ & + \frac{-1}{4\rho\pi\omega^2}\int_{-\infty}^{\infty}\mathrm{Re}(-\gamma - k^2/\nu - \gamma\mathrm{e}^{-\mathrm{i}\gamma z_0}C_2 - k\mathrm{e}^{-\mathrm{i}\nu z_0}D_2)\,\mathrm{d}k,\end{aligned} \tag{38}$$

where $\mathbf{x} = (0, z_0)^{\mathrm{T}}$, $\nu = \sqrt{(\omega/\beta)^2 - k^2}$ and $\gamma = \sqrt{(\omega/\alpha)^2 - k^2}$, with $\mathrm{Im}\,\gamma < 0$. $\nu$ and $\gamma$ are the vertical wavenumbers of S and P waves, respectively, and

$$\left.\begin{aligned} & A_2 = \{(4k^2\nu\gamma - (k^2-\nu^2)^2)A_1 - 4k\nu(k^2-\nu^2)B_1\}F^{-1}(k,\omega), \\ & B_2 = \{4k\gamma(k^2-\nu^2)A_1 + (4k^2\nu\gamma - (k^2-\nu^2)^2)B_1\}F^{-1}(k,\omega), \\ & A_1 = k\gamma^{-1}\mathrm{e}^{-\mathrm{i}\gamma z_0}, \quad B_1 = \mathrm{e}^{-\mathrm{i}\nu z_0}, \\ & C_2 = \{(4k^2\nu\gamma - (k^2-\nu^2)^2)C_1 - 4k\nu(k^2-\nu^2)D_1\}F^{-1}(k,\omega), \\ & D_2 = \{4k\gamma(k^2-\nu^2)C_1 + (4k^2\nu\gamma - (k^2-\nu^2)^2)D_1\}F^{-1}(k,\omega), \\ & C_1 = -\mathrm{e}^{-\mathrm{i}\gamma z_0} \quad \text{and} \quad D_1 = k\nu^{-1}\mathrm{e}^{-\mathrm{i}\nu z_0}, \end{aligned}\right\} \tag{39}$$

where $F(k,\omega) = (k^2 - \nu^2)^2 + 4k^2\nu\gamma =$ Rayleigh function. We computed these integrals numerically (from $-\omega/\beta$ to $\omega/\beta$ in order to have real kernels). The contributions from the Rayleigh pole, which are obtained using the residue at $k = k_R = \omega/c_R$ (see [28]), are given by

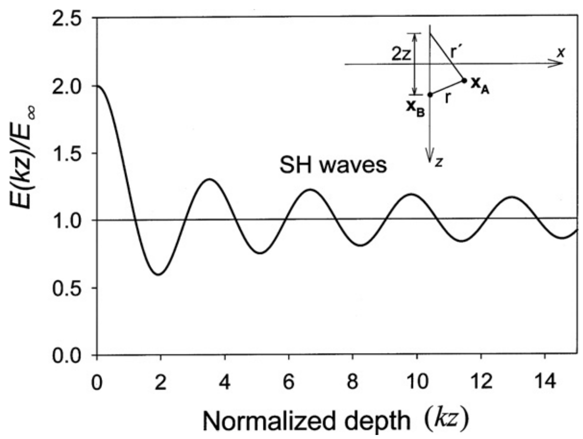


Fig. 3. Normalized energy density for a diffuse SH wave field in a 2D half-space.

$$\mathrm{Im}\left[G_{33}^{\mathrm{R}}(z_0,z_0)\right] = \frac{-1}{2\rho\omega^2}\mathrm{Re}\left\{\frac{-\mathrm{i}\gamma e^{-\mathrm{i}\gamma z_0}C_2 - \mathrm{i}k e^{-\mathrm{i}\nu z_0}D_2}{F'(k_{\mathrm{R}},\omega)}\right\} \text{ and} \qquad (40)$$

$$\mathrm{Im}\left[G_{11}^{\mathrm{R}}(z_0,z_0)\right] = 0.$$

For a Poisson solid ($\lambda = \mu$) the energy ratio $E(z/\Lambda_{\mathrm{R}})/E_\infty = -4\mu E_{\mathrm{S}}\mathrm{Im}[G_{mm}(z,z)]/E_\infty$ is shown in Fig. 4 against the normalized depth $z/\Lambda_{\mathrm{R}}$ where $\Lambda_{\mathrm{R}}$ is the wavelength of Rayleigh waves. Note that $E_\infty = E_{\mathrm{S}} + E_{\mathrm{P}} = E_{\mathrm{S}} \times (1 + \beta^2/\alpha^2)$. The contributions from $G_{11}$ and $G_{33}$ are pointed out in the plot. They correspond to the horizontal and vertical degrees of freedom and for increasing depth they tend to 1/2 in agreement with the equipartition theory.

On the other hand, the free surface influence on energy densities can be obtained from the first part of Eq. (33) assuming isotropic illumination of P and S waves from the deep half-space (see [15,17]). Energy density against depth was obtained by cross-correlating equipartitioned plane P and S waves, including their free surface reflections, and Rayleigh waves. The averages for P and S wave energies are calculated using incoming homogeneous waves with incidence angles from $-\pi/2$ to $\pi/2$ and the integrals are transformed to the horizontal wavenumber domain using appropriate changes of variable. The details are cumbersome and will be presented and discussed elsewhere. Fig. 5 depicts the normalized energy densities associated with incident P, S and Rayleigh waves for a Poisson solid ($\lambda = \mu$). The normalized total energy $E(z,\omega)/E_\infty$ tends to one, while at depth the ratio $E_{\mathrm{S}}/E_{\mathrm{P}}$ stabilizes to the theoretical value $\alpha^2/\beta^2$. At the surface the total energy density is twice the value for deep space. Oscillations associated with the depth dependence of the S to P wave energy ratios were also observed by Hennino et al. [4].

### 5.3. The 2D cylindrical inhomogeneity

The 2D Green function for a medium with a cylindrical inhomogeneity is a canonical problem that has been studied in the context of the Green function retrieval. If the elastic space is isotropically illuminated with plane waves, the identity of Eq. (31) is confirmed [17]. The complete analytical solution was developed for the anti-plane SH case and for the in-plane P–SV waves as well. The energy densities of the isotropic illumination given by P and SV waves are such that the ratio $E_{\mathrm{S}}/E_{\mathrm{P}} = \alpha^2/\beta^2$. The Fourier transform of the azimuthal average of the cross-correlation of the motion between two points within an elastic medium is in that case proportional to the imaginary part of the exact Green tensor function between these points [17].

For purposes of illustration let us consider the simplest SH case. For a cavity of radius $a$, the 2D anti-plane Green function can be expressed using Graf's addition theorem (see [29]) as the sum of incident and diffracted waves:

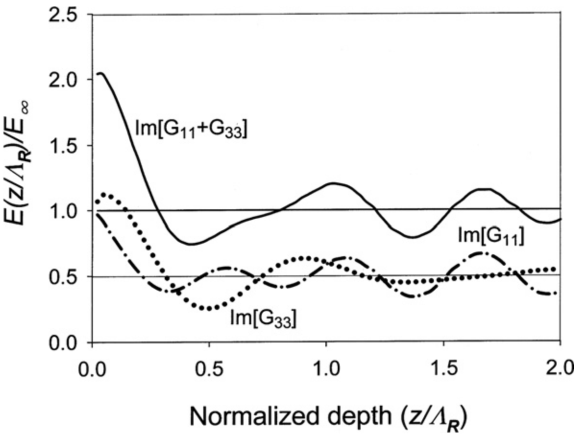


Fig. 4. Normalized energy density against normalized depth for a diffuse vector wave field in a 2D half-space, considering the degrees of freedom. Continuous line shows the total contribution of the two degrees of freedom (see Eq. (33)) and is represented by $\mathrm{Im}[G_{11} + G_{33}]$. Dash-dot line represents the part of horizontal motion which was computed from $\mathrm{Im}[G_{11}]$. Dotted line depicts the participation of vertical displacements obtained from $\mathrm{Im}[G_{33}]$. The partition factors tend, as the depth increases, to the theoretical value of one half each.

$$G_{22}(\mathbf{x},\mathbf{y};\omega) = v^0 + v^d = \frac{1}{4i\mu}\left\{H_0^{(2)}(kR) + \sum_{n=0}^{\infty} \varepsilon_n A_n H_n^{(2)}(kd) H_n^{(2)}(kr)\cos n\theta\right\}, \tag{41}$$

where $H_n^{(2)}(\cdot)$ = Hankel function of the second kind and order $n$, $k = \omega/\beta$ = shear wavenumber, $R = |\mathbf{x} - \mathbf{y}|$ = distance between source and receiver (see Fig. 6), $d$ = distance of $\mathbf{y}$ to the center of cavity, $r$ and $\theta$ = cylindrical coordinates and $\varepsilon_n$ = Neumann factor (=1 if $n = 0$ or =2, otherwise).

The coefficients $A_n$ of the expansion of the diffracted field are obtained from the boundary conditions of a traction-free boundary at $r = a$; this gives

$$A_n = -\frac{J_n'(ka)}{H_n^{(2)'}(ka)}. \tag{42}$$

As in Eq. (35), the energy density as function of frequency, distance $d$ to the center of the cavity and its radius $a$ is

$$E = E_\infty \times \left(1 + \sum_{n=0}^{\infty} \varepsilon_n \left[\frac{J_n'^2(ka)(J_n^2(kd) - Y_n^2(kd)) + 2J_n'(ka)Y_n'(ka)J_n(kd)Y_n(kd)}{J_n'^2(ka) + Y_n'^2(ka)}\right]\right) \tag{43}$$

In Fig. 7, the ratio $E(d,\omega,a)/E_\infty$ is depicted against the normalized (radial) distance to the cavity edge $k(d - a)$ for various values of the ratio $\Lambda/a = 2\pi/ka$ of the shear wavelength and cavity radius $a$ in order to observe the relative effect of the size of the cavity. In fact, the various lines correspond to $\Lambda/a = 2\pi/ka = 20$, 5 and 0.25,

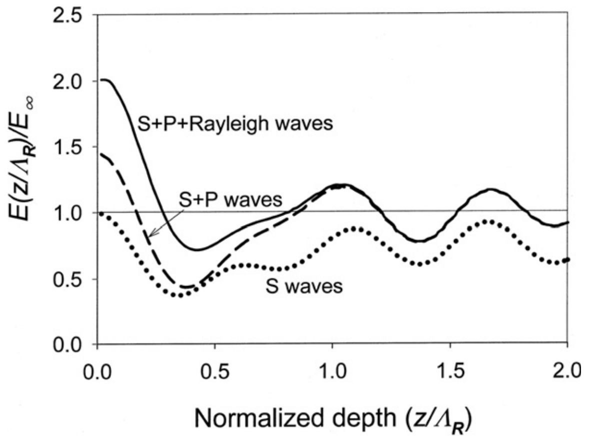


Fig. 5. Normalized energy density, for a diffuse vector wave field in a 2D half-space, expressed with regard to incoming wave types. Continuous line shows the total S, P and Rayleigh wave contributions. Dotted line represents the contribution of incoming plane S waves and its reflections, while dashed line exhibits the additional contribution due to the incoming P waves and its reflections. The energy density level of the dashed line is in fact due to incident and reflected plane waves. The part of Rayleigh waves is, as expected, limited within a strip near the surface and practically disappears for a depth equal to the Rayleigh wavelength.

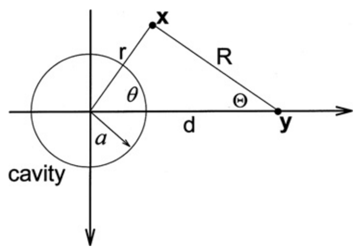


Fig. 6. Cylindrical cavity of radius $a$ within the infinite space E. The line SH source is applied at $\mathbf{y}$ while the receiver is at $\mathbf{x}$ ($r$, $\theta$ in polar coordinates). The distance between source and receiver is $R$.

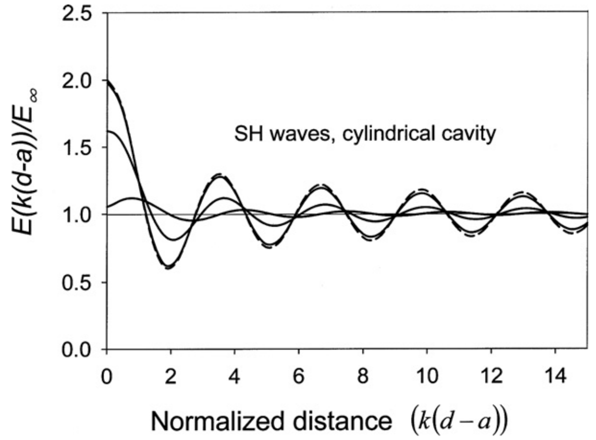


Fig. 7. Normalized energy density for a diffuse SH wave field in a 2D half-space containing a cavity with radius $a$. The various lines correspond to $\Lambda/a = 2\pi/ka = 20$, 5 and 0.25, respectively, the value 0.0 is plotted with dashed line and represents the half-space result as shown in Fig. 3.

respectively. For wavelengths shorter than about one fourth the radius $a$, the fluctuations are practically equivalent to those of the half-space. In the figure, the zero value of this ratio is plotted with a dashed line and represents the half-space.

## 6. Conclusions

Assuming the homogeneous elastic case and a set of uncorrelated boundary sources, a correlation type representation theorem allowed us to find that the resulting elastic field radiated by the far away boundary is equipartitioned. This implies that in infinite space the ratio of P to S energies stabilizes to a constant value. If a homogeneous envelope is considered, the representation theorem allows the verification of the well known result that by averaging correlations of motion produced by a diffuse field (such as that of coda waves) it is possible to retrieve the Green function which is of interest for traditional imaging.

In both the homogeneous and heterogeneous cases, the equipartition of the background field is a necessary and sufficient condition to retrieve the *exact* Green function from correlations. We show that the trace of the imaginary part of the Green function tensor at the source is proportional to the energy density. Thus, the Green function availability may allow the establishing of the theoretical energy density of a diffuse field generated by a background equipartitioned field. This may be useful to assess the diffuse nature of seismic ground motion (from a limited set of observation points).

The cases of a full homogeneous elastic space, a space with an elastic cylindrical cavity and a half-space were studied. Energy densities of both P and S waves and their ratios were found to exhibit fluctuations in space and frequency that are due to nearby edges and/or scatterers. These fluctuations may be useful for exploration purposes. The analysis of the spatial and frequency variations may help to detect underground features. Such a method would not rely on specific sources but would take advantage of the apparently random nature of diffuse fields.

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## References

[1] K. Aki, Space and time spectra of stationary stochastic waves with special reference to microtremors, Bull. Earthq. Res. Inst. 35 (1957) 415–456.
[2] H. Sato, M. Fehler, Wave Propagation and Scattering in the Heterogeneous Earth, Springer-Verlag, New York, 1998.
[3] N.M. Shapiro, M. Campillo, L. Margerin, S.K. Singh, V. Kostoglodov, J. Pacheco, The energy partitioning between P and S waves and the diffusive character of the seismic coda, Bull. Seismol. Soc. Am. 90 (2000) 655–665.
[4] R. Hennino, N. Trégourès, N.M. Shapiro, L. Margerin, M. Campillo, B. van Tiggelen, R.L. Weaver, Observation of equipartition of seismic waves in Mexico, Phys. Rev. Lett. 86 (15) (2001) 3447–3450.
[5] K. Aki, B. Chouet, Origin of coda waves: source, attenuation and scattering effects, J. Geophys. Res. 80 (1975) 3322–3342.
[6] R.L. Weaver, On diffuse waves in solid media, J. Acoust. Soc. Am. 71 (1982) 1608–1609.
[7] R. Snieder, Coda wave interferometry and the equilibration of energy in elastic media, Phys. Rev. E 66 (2002) 04615-1-8.
[8] L.V. Ryzhik, G.C. Papanicolau, J.B. Keller, Transport equations for elastic and other waves in random media, Wave Motion 24 (1996) 327–370.
[9] M. Campillo, A. Paul, Long range correlations in the seismic coda, Science 299 (2003) 547–549.
[10] N.M. Shapiro, M. Campillo, Emergence of broadband Rayleigh waves from correlations of the ambient seismic noise, Geophys. Res. Lett. 31 (2004) L07614, doi:10.1029/ 2004GL019491.
[11] K. Sabra, G.P. Gerstoft, P. Roux, W.A. Kuperman, M.C. Fehler, Extracting time-domain Green's function estimates from ambient seismic noise, Geophys. Res. Lett. 32 (2005) L03310, doi:10.1029/2004GL021862.
[12] M. Campillo, Phase and correlation in 'random' seismic fields and the reconstruction of the Green function, Pure Appl. Geophys. 163 (2006) 475–502.
[13] R. Snieder, Extracting the Green's function from the correlation of coda waves: a derivation based on stationary phase, Phys. Rev. E 69 (2004) 046610.
[14] A.E. Malcolm, J.A. Scales, B.A. van Tiggelen, Extracting the Green function from diffuse, equipartitioned waves, Phys. Rev. E 70 (2004) 015601(R).
[15] F.J. Sánchez-Sesma, M. Campillo, Retrieval of the Green function from cross-correlation: the canonical elastic problem, Bull. Seismol. Soc. Am. 96 (2006) 1182–1191.
[16] R.L. Weaver, O.I. Lobkis, Diffuse fields in open systems and the emergence of the Green's function, J. Acoust. Soc. Am. 116 (2004) 2731–2734.
[17] F.J. Sánchez-Sesma, J.A. Pérez-Ruiz, M. Campillo, F. Luzón, The elastodynamic 2D Green function retrieval from cross-correlation: the canonical inclusion problem, Geophys. Res. Lett. 33 (2006) L13305, doi:10.1029/2006GL026454.
[18] R.L. Weaver, Diffuse elastic waves at a free surface, J. Acoust. Soc. Am. 78 (1985) 131–136.
[19] K. Wapenaar, Retrieving the elastodynamic Green's function of an arbitrary inhomogeneous medium by cross correlation, Phys. Rev. Lett. 93 (2004) 254301-1–254301-4.
[20] D.-J. van Manen, A. Curtis, J.O.A. Robertsson, Interferometric modelling of wave propagation in inhomogeneous elastic media using time-reversal and reciprocity, Geophysics 71 (2006) S147–S160, doi:10.1190/1.221321.
[21] M. Fink, Time reversal of acoustic waves, Phys. Today 50 (1997) 34–40.
[22] A. Derode, E. Larose, M. Tanter, J. Rosny, A. Tourin, M. Campillo, M. Fink, Recovering the Green's function from field–field correlations in an open scattering medium, J. Acoust. Soc. Am. 113 (2003) 2973–2976.
[23] O.A. Godin, Recovering the acoustic Green's function from ambient noise cross correlation in an inhomogeneous moving medium, Phys. Rev. Lett. 97 (2006) 054301-1-4.
[24] R. Snieder, Retrieving the Green's function of the diffusion equation from the response to a random forcing, Phys. Rev. E. 74 (2006) 046620.
[25] A. Paul, M. Campillo, L. Margerin, E. Larose, A. Derode, Empirical synthesis of time-asymmetrical Green function from the correlation of coda waves, J. Geophys. Res. 110 (2005), doi:10.1039/2004JB003521.
[26] F.J. Sánchez-Sesma, M. Campillo, Diffraction of P, SV and Rayleigh waves by topographic features: a boundary integral formulation, Bull. Seismol. Soc. Am. 81 (1991) 2234–2253.
[27] F.J. Sánchez-Sesma, F. Luzón, Seismic response of three-dimensional alluvial valleys for incident P, S and Rayleigh waves, Bull. Seismol. Soc. Am. 85 (1995) 269–284.
[28] W.M. Ewing, W.S. Jardetzky, F. Press, Elastic Waves in Layered Media, McGraw-Hill Book Co., New York, 1957.
[29] M. Abramowitz, A. Stegun, Handbook of Mathematical Functions, Dover Publications Inc., New York, 1972.
[30] P. Roux, W.A. Kuperman, W.S. Hodgkiss, H.C. Song, T. Akal, M. Stevenson, A nonreciprocal implementation of time reversal in the ocean, J. Acoust. Soc. Am. 116 (2004) 1009–1015.
[31] R.L. Weaver, Ward identities and the retrieval of Green's functions in the correlations of a diffuse field, Wave Motion 45 (2008) 427–435.