# Finite Difference Method (FDM)

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# A global vision

- *Differential Calculus* (Newton, 1687 & Leibniz 1684)
  - Find solutions of a differential equation (DE) of a dynamic system.
- *Chaos Systems* (Poincaré, 1881)
  - Find properties of solutions of the DE of a dynamic system.
- *Chaos & Stability* (Smale, 1960)
  - Find properties of solutions of a physical system without knowing its DE

After the presentation of Etienne Ghys, 13 october 2009

*This course is about the differential calculus using the finite difference approach familiar to Newton & Leibniz*

# Bibliography on Finite Difference Methods :


**A. Taflove and S. C. Hagness: Computational Electrodynamics: The Finite-Difference Time-Domain Method, Third Edition, Artech House Publishers, 2005**

**O.C. Zienkiewicz and K. Morgan: Finite elements and approximmation, Wiley, New York, 1982**

**W.H. Press et al, Numerical recipes in FORTRAN/C … Cambridge University Press, USA, 20XX …**


**http://en.wikipedia.org/wiki/Finite-difference_time-domain_method**

**http://en.wikipedia.org/wiki/Upwind_scheme**

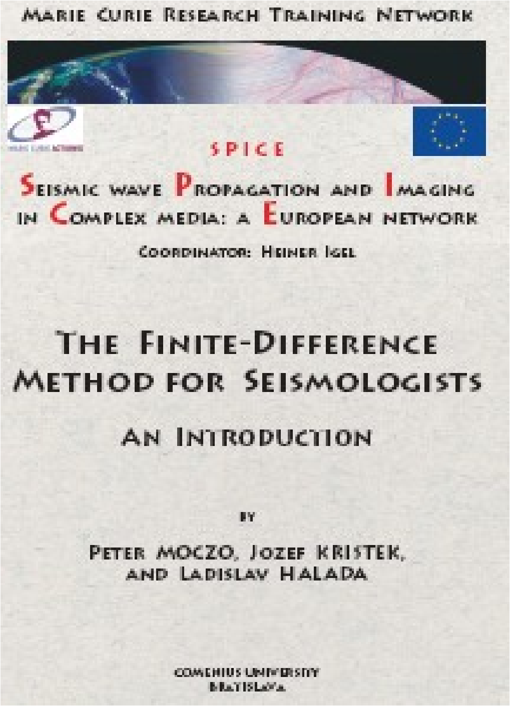
**http://en.wikipedia.org/wiki/Lax-Wendroff_method**

Spice group in Europe : http://www.spice-rtn.org

FDTD introduction :

ftp://ftp.seismology.sk/pub/papers/FDM-Intro-SPICE.pdf

By P. Moczo, J. Kristek and L. Halada



# Why the FDM ?

- Approximation of derivatives

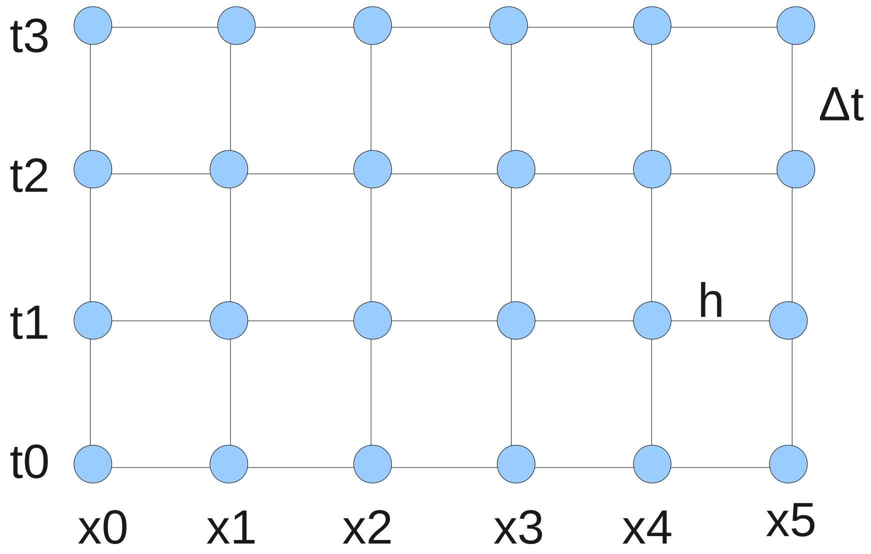
⇒ numerical solving of differential equations

- One of the most important numerical methods in seismology, EQ ground motion modeling (risk) and seismic (exploration)

# What is the principle ?

- Construction of a ***discrete*** finite-difference model of the problem

  - Coverage of the computational domain by a *space-time grid*



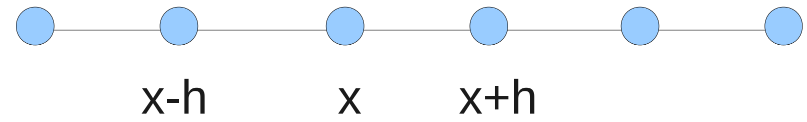

*Δt = time step,   h = grid spacing*

  - Approx. to derivatives and initial conditions at the grid points
  - Boundaries conditions at the end points
  - Construction of a system of the finite-difference equations

# Some examples

- $$\frac{\partial^2 u}{\partial^2 t} = c^2 \frac{\partial^2 u}{\partial^2 x}$$ Wave (seismology, GPR)
- $$\frac{\partial u}{\partial t} = \frac{1}{\kappa} \frac{\partial^2 u}{\partial x^2}$$ Diffusion (EM31,geothermal science, magnetotelluric)
- $$0 = \frac{\partial^2 u}{\partial x^2}$$ Potential (electric,magnetic, gravimetry)

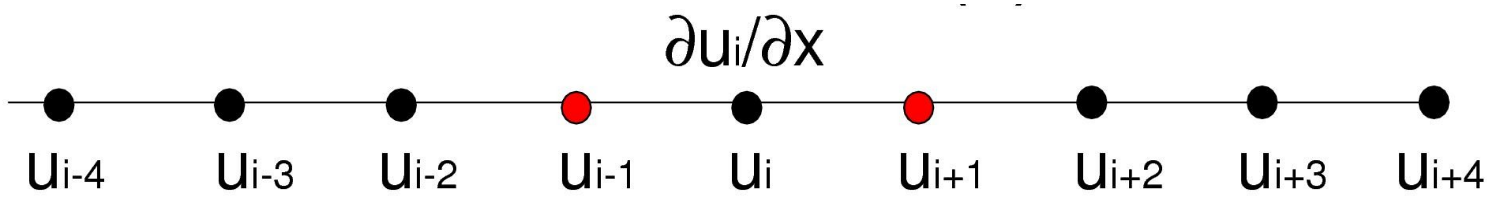
# Derivative approximation



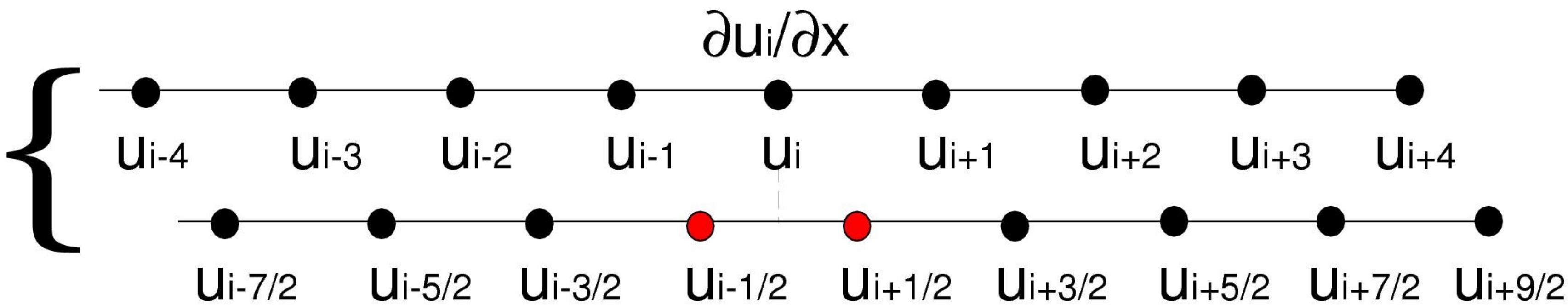

- $1^{st}$ derivative
  - Forward : $$u'(x) = \frac{u(x+h) - u(x)}{h}$$
  - Backward : $$u'(x) = \frac{u(x) - u(x-h)}{h}$$
  - Centered : $$u'(x) = \frac{u\left(x + \frac{1}{2}h\right) - u\left(x - \frac{1}{2}h\right)}{h}$$

$$\Longleftrightarrow \quad u'(x) = \frac{u(x+h) - u(x-h)}{2h}$$

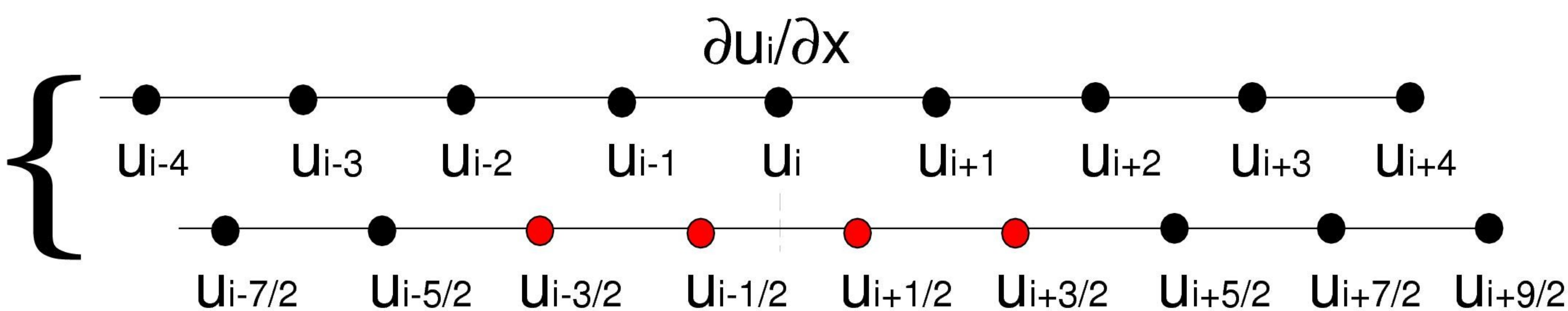
**Second-order accurate central-difference approximation**




**Leapfrog second-order accurate central-difference approximation**




**Leapfrog 4th-order accurate central-difference approximation**



# Derivative approximation

- 2nd derivative
  - By differentiating the 1st derivative
  - By using the Taylor expansion

# Derivative approximation ($2^{nd}$ derivative)

- Taylor expansion *(uniform discretisation on the domain)*

$$u(x_i+h)_n = u_{i+1,n} = u_{i,n} + h\frac{\partial u}{\partial x_{i,n}} + \frac{h^2}{2}\frac{\partial^2 u}{\partial^2 x_{i,n}} + \frac{h^3}{6}\frac{\partial^3 u}{\partial^3 x_{i,n}} + \frac{h^4}{24}\frac{\partial^4 u}{\partial^4 x_{i,n}}$$

$$+$$

$$u(x_i-h)_n = u_{i-1,n} = u_{i,n} - h\frac{\partial u}{\partial x_{i,n}} + \frac{h^2}{2}\frac{\partial^2 u}{\partial^2 x_{i,n}} - \frac{h^3}{6}\frac{\partial^3 u}{\partial^3 x_{i,n}} + \frac{h^4}{24}\frac{\partial^4 u}{\partial^4 x_{i,n}}$$

$$\Downarrow$$

$$u_{i-1,n} + u_{i+1,n} = 2\mathrm{u}_{i,n} + h^2\frac{\partial^2 u}{\partial^2 x_{i,n}} + \frac{h^4}{12}\frac{\partial^4 u}{\partial^4 x_{i,n}}$$

D'où

$$\frac{\partial^2 u}{\partial^2 x_{i,n}} = \frac{u_{i+1,n} + u_{i-1,n} - 2\mathrm{u}_{i,n}}{h^2} + O(h^2)$$

*$i$ = index of space*
*$n$ = index of time*
*$h$ = space step*

# Example : cosinus (1D)

1) Define the space domain

$x = 0 : 0.5 : 4\pi$ → (h=0.5)

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$$(\cos(x))' = \frac{\cos(x+h) - \cos(x-h)}{2h}$$

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x = 0 : 0.5 : 4π → (h=0.5)

2) Approximate the derivative

$$(\cos(x))' = \frac{\cos(x+h) - \cos(x-h)}{2\mathrm{h}}$$

3) Verification of the convergence

We know that (cos(x))' = -sin(x)

Does $\frac{\cos(x+h) - \cos(x-h)}{2\mathrm{h}} - \sin(x) \rightarrow 0$ when h smaller ?

# Wave equation

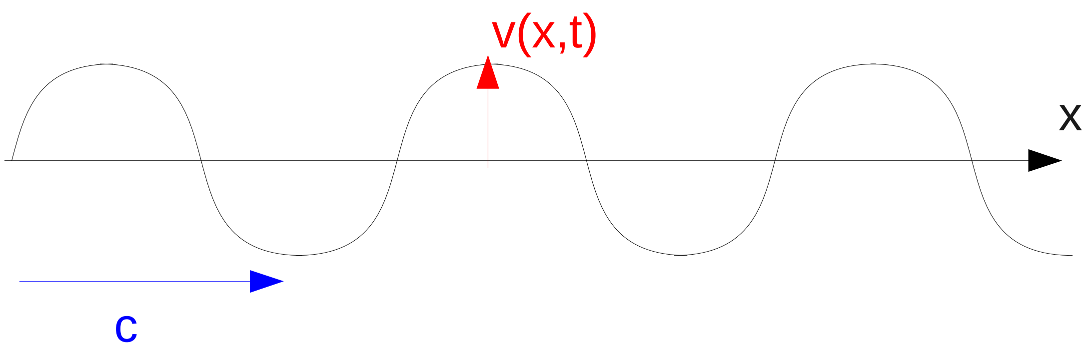
$$\frac{\partial^2 u}{\partial^2 t} = c^2 \frac{\partial^2 u}{\partial^2 x}$$

- c = wave speed
- u(x,t) = displacement of the particle
  - At point ***x***
  - At time ***t***

# Wave propagation on a string

- $u(x,t)$ = displacement at a point x of a string at the time t
- $\sigma(x,t)$ = stress at this same point
- $v(x,t)$ = displacement velocity at this same point

$v(x,t) \neq c$



$u(x,t)$, $\sigma(x,t)$, $v(x,t)$

***How to find these quantities when the string properties are varying ???***

# Let's define other variables

- *Why ??* to reduce the order of the derivatives

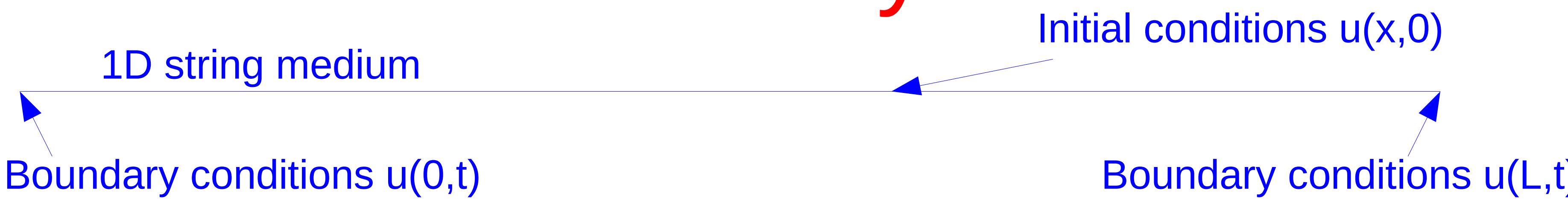
  to introduce some physics (stress)

$$\sigma = E\frac{\partial u}{\partial x} \qquad v = \frac{\partial u}{\partial t}$$

$$\begin{cases} \dfrac{\partial v}{\partial t} = \dfrac{1}{\rho}\dfrac{\partial \sigma}{\partial x} \\ \dfrac{\partial \sigma}{\partial t} = E\dfrac{\partial v}{\partial x} \end{cases}$$

*Mechanical equation without external force*

# Initial and boundary conditions




- Initial conditions : at time t=0
  - v(x,0)=0
  - σ(x,0)=0
- Boundary conditions
  - *Free surface* : σ(0,t) = 0

    σ(L,t) = 0
  - *Rigid surface* : v(0,t) = 0

    v(L,t) = 0
  - *Reflection/transmission*

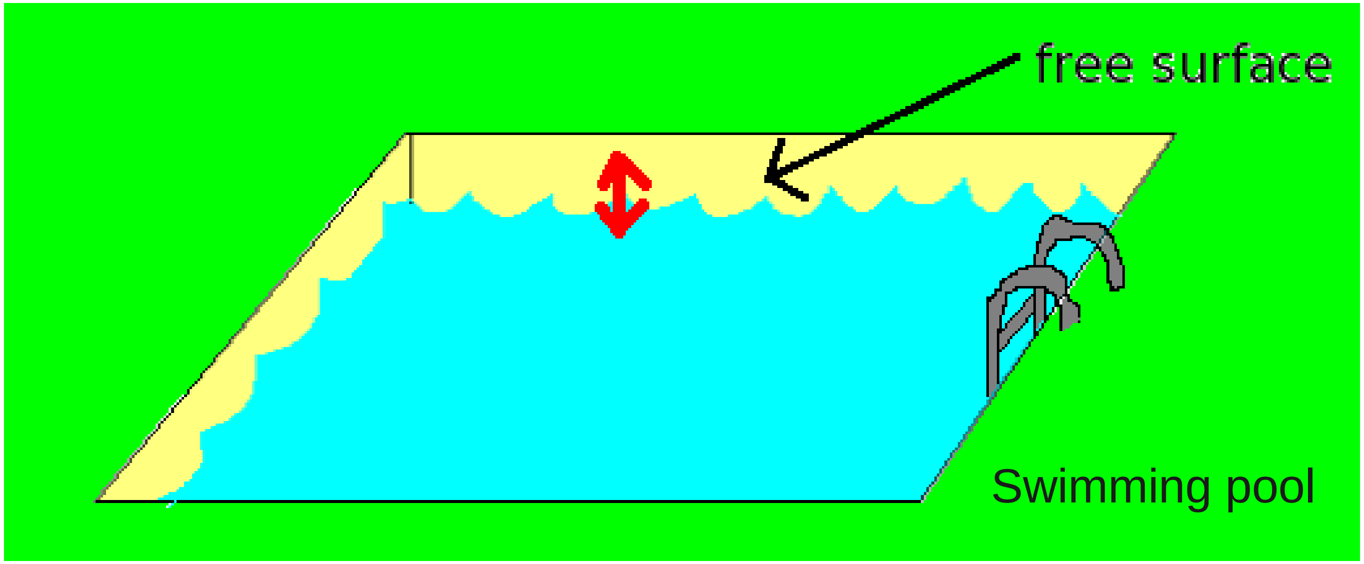
    (in energy) Z=ρc

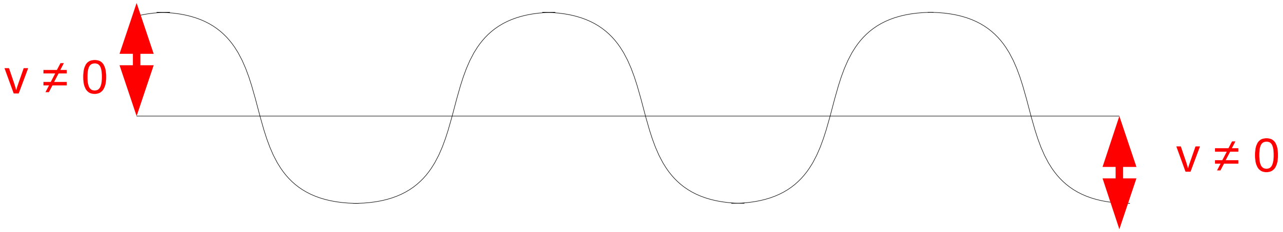
    1 : space 1 2 : space 2

$$R = \frac{Z_1 - Z_2}{Z_1 + Z_2} \qquad T = \frac{2 Z_1}{Z_1 + Z_2}$$

# Boundaries conditions (1)

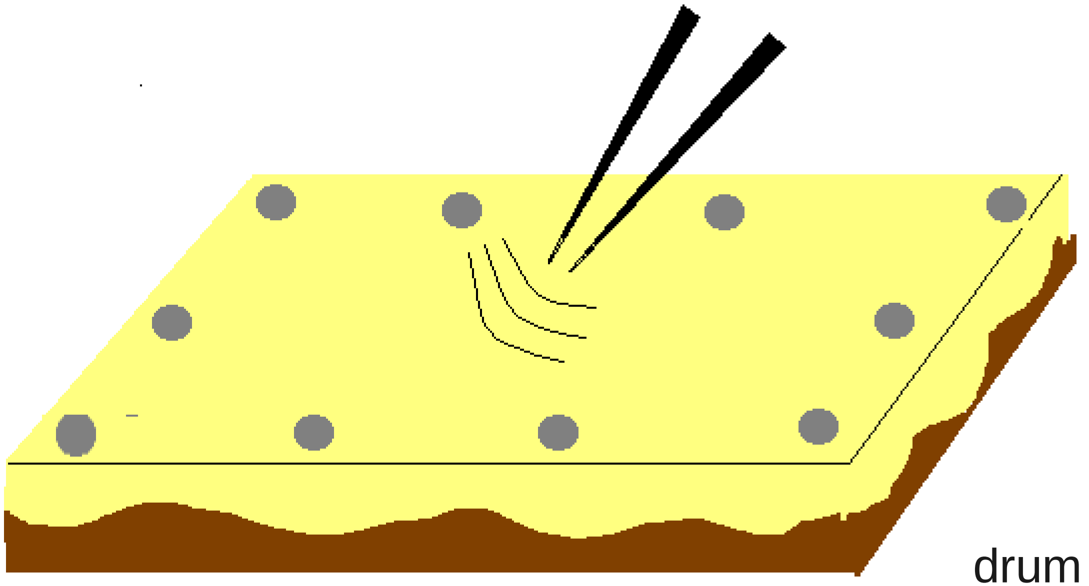
- *Free surface* : we allow motion ( v ≠ 0)

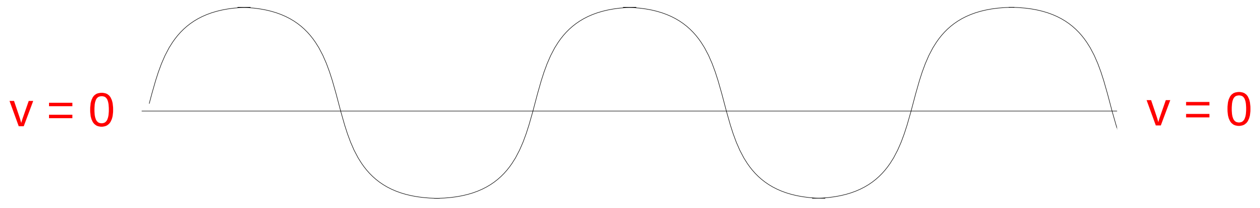


# Boundaries conditions (2)

- *Rigid surface* : no motion



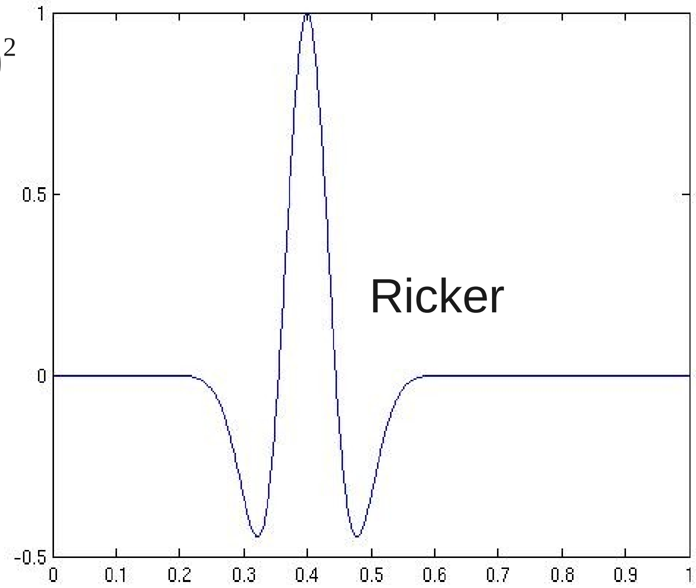



# Source excitation

- Impulsive source :
  - You have to add a term at the equation :

$$\frac{\partial^2 u}{\partial^2 t} = c^2 \frac{\partial^2 u}{\partial^2 x} \boxed{+s}$$

  - Ricker : $s = \left(1 - 2\pi^2 f^2 (t - t_0)^2\right) e^{-\pi^2 f^2 (t - t_0)^2}$



# Source excitation

- **Impulsive source :**
  - You have to add a term at the equation :

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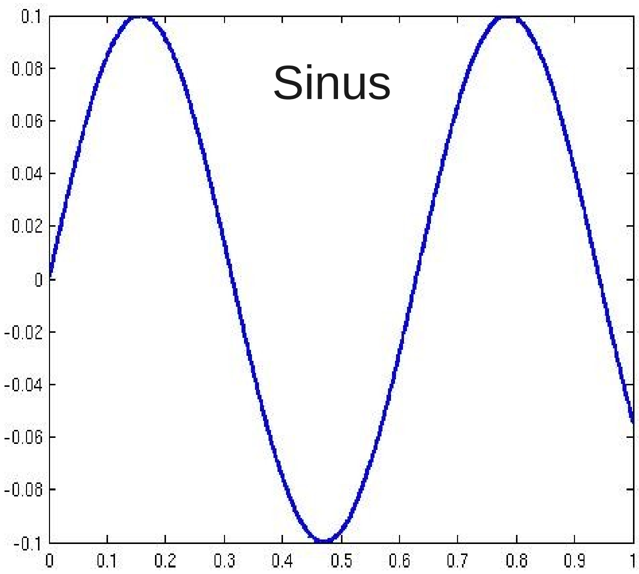
  - *Ricker* : $s = (1 - 2\pi^2 f^2 t^2) e^{-\pi^2 f^2 t^2}$

ou

- **Oscillatory source**
  - *Sinus* : $s = \frac{\sin(2\mathrm{ft})}{2\mathrm{f}}$

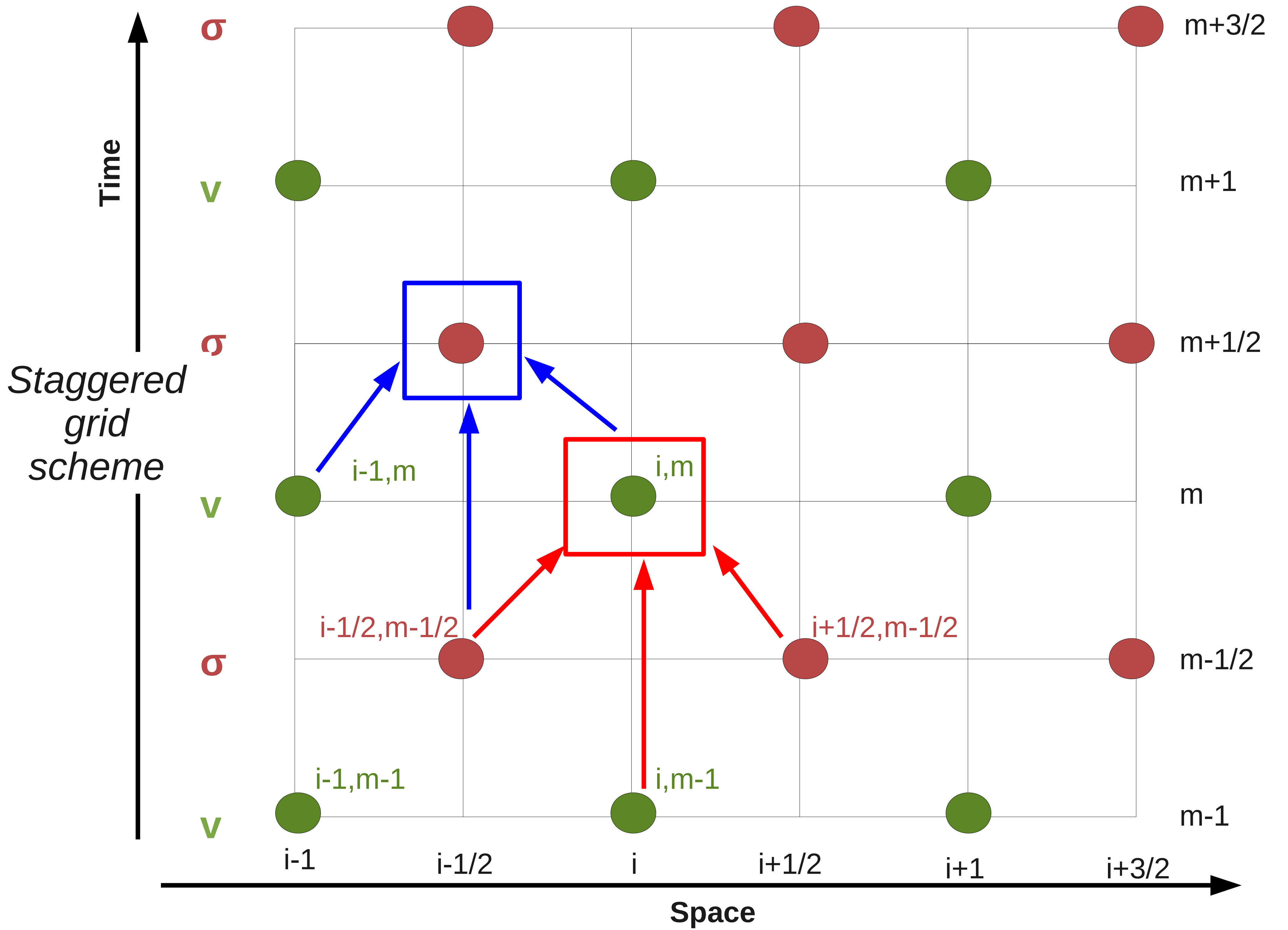


# Source radiation

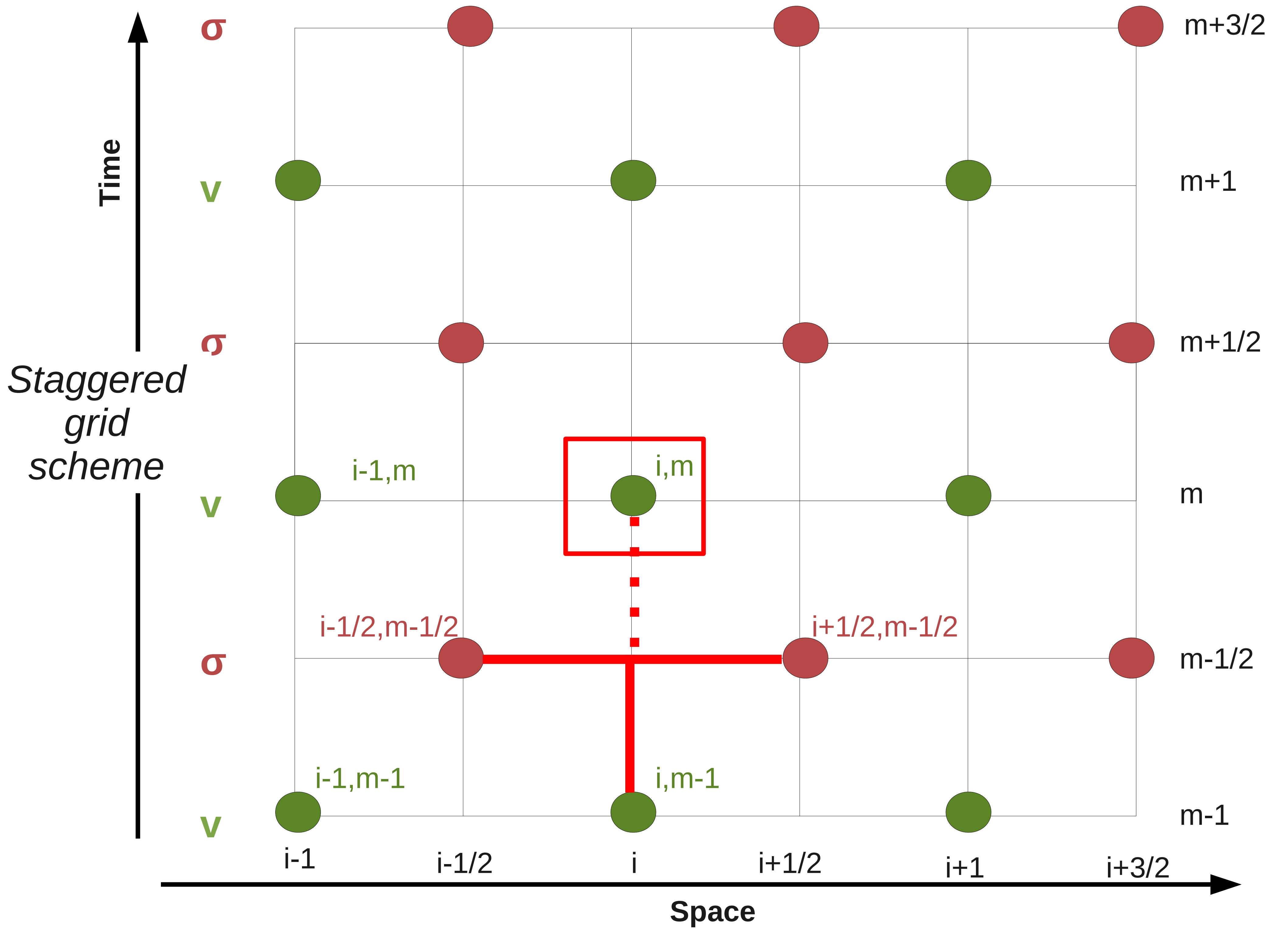
- **Directional source (hammer) :**
  - f(z)

- **Explosive source :**
  - Application of opposite sign forces on two nodes or a fictious force between two nodes

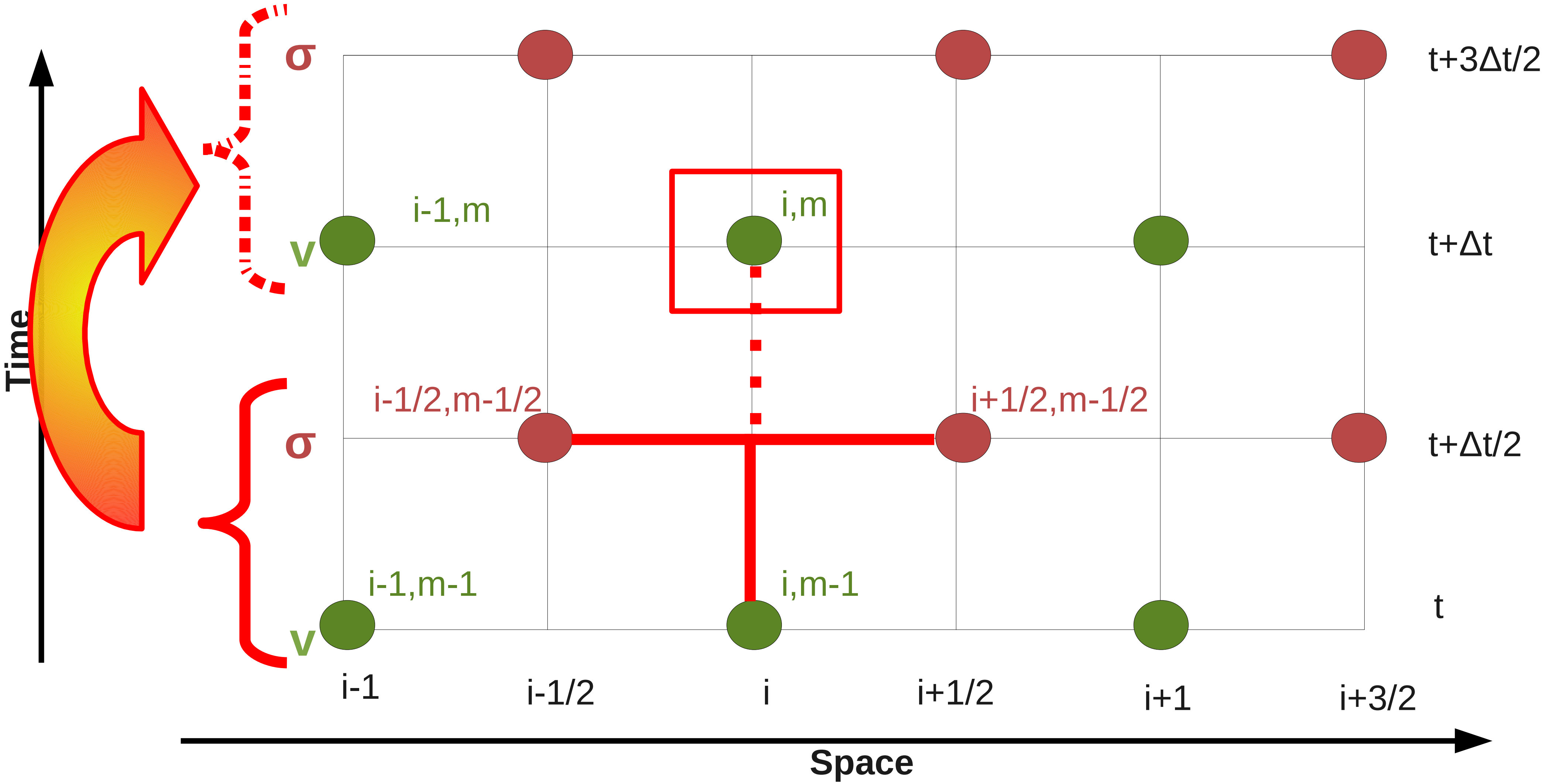
# How to discretize the problem ??



Time
Staggered grid scheme
σ
v
σ
v
σ
v
m+3/2
m+1
m+1/2
m
m-1/2
m-1
i-1,m
i,m
i-1/2,m-1/2
i+1/2,m-1/2
i-1,m-1
i,m-1
i-1
i-1/2
i
i+1/2
i+1
i+3/2
Space



Time
Staggered grid scheme
σ
v
σ
v
σ
v
m+3/2
m+1
m+1/2
m
m-1/2
m-1
i-1,m
i,m
i-1/2,m-1/2
i+1/2,m-1/2
i-1,m-1
i,m-1
i-1
i-1/2
i
i+1/2
i+1
i+3/2
Space




We always keep 2 lines :

We know the 2 previous lines, (t and t+Δt/2)
We are looking for the two next (t+Δt and t+3Δt/2)

# How to discretize the problem ??

(x,t) grid with space step h and time step Δt

$$
\begin{cases}
\dfrac{v_{i,m}-v_{i,m-1}}{\Delta t}=b_i\dfrac{\sigma_{i+1/2,m-1/2}-\sigma_{i-1/2,m-1/2}}{h} \\[2ex]
\dfrac{\sigma_{i+1/2,m+1/2}-\sigma_{i+1/2,m-1/2}}{\Delta t}=E_{i+1/2}\dfrac{v_{i+1,m}-v_{i,m}}{h}
\end{cases}
$$

$$
\begin{cases}
v_{i,m}=v_{i,m-1}+b_i\dfrac{\Delta t}{h}\left(\sigma_{i+1/2,m-1/2}-\sigma_{i-1/2,m-1/2}\right) \\[2ex]
\sigma_{i+1/2,m+1/2}=\sigma_{i+1/2,m-1/2}+E_{i+1/2}\dfrac{\Delta t}{h}\left(v_{i+1,m}-v_{i,m}\right)
\end{cases}
$$

# All you need is there

- **Loop over time**   k=1,n_max   t=(k-1)*dt
- **Loop over velocity field**

  i=1,i_max   x=(i-1)*dx

  compute velocity field from stress field

  apply velocity boundary conditions

  end
- **Loop over stress field**

  i=1,i_max   x=(i-1)*dx

  compute stress field from velocity field

  apply stress boundary conditions

  end
- **Set external source effect   replacing or adding external values at specific points**
- End loop over time

```
Do l=1,Nsources                                          !loop over sources
    Do i= 1,Nx
    v(i) = 0 ;  σ(i) = 0                                 !Initial conditions
    End do
    Do n=1,Nt                                            !loop over time steps
       Update f(n)
       Do i=1,Nx                                         !loop over spatial steps
          v(i) = v(i)+(b(i).dt/h)[σ(i+1/2)-σ(i-1/2)]      !In-place update of v
       End do
       Implementation of boundary condition for v at t=(n+1)dt
       v(is) = v(is)+f(n)                                !Application of source
       Do i=1,Nx                                         !Loop over spatial steps
          σ(i+1/2) = σ(i+1/2)+(1/E(i+1/2).dt/h)[v(i+1)-v(i)]   !In-place update of σ
       End do
       Implementation of boundary condition for σ at t=(n+3/2)dt
       Write v at t=(n+1)dt and σ at t=(n+3/2)dt
    End do
End do
```

Algorithm

Rq : the velocity and stress fields are stored in core only at 1 time