Scattering control without knowing the wave speed

– inverse problem for the wave equation with piecewise smooth wave speeds

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multiple scattering, unknown piecewise smooth wave speed

disentangling multiple scattering

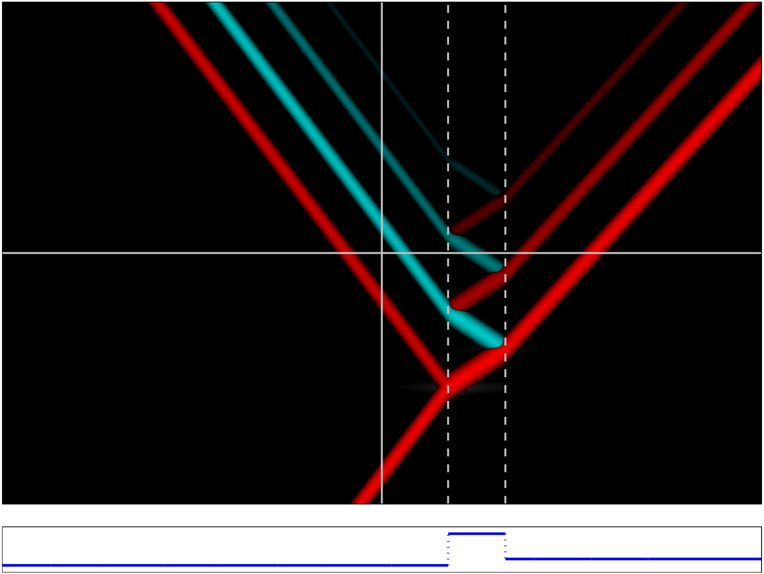
- instantaneous time mirrors in an extended subsurface
- scattering control, detection of kinetic energy, projections

extensive (composite) data manipulations

inverse problem

- broken boundary normal ('time') coordinates
- wave-based coordinate transformation reconstruction from partial data $\rightarrow$ wave speed, discontinuities
- interface detection without the wave speed

controlling multiple scattering — dimension 1



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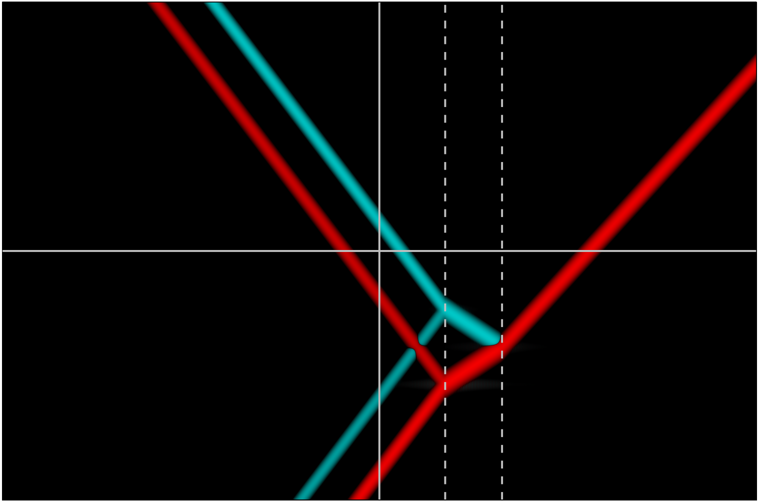

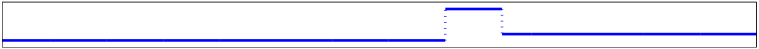

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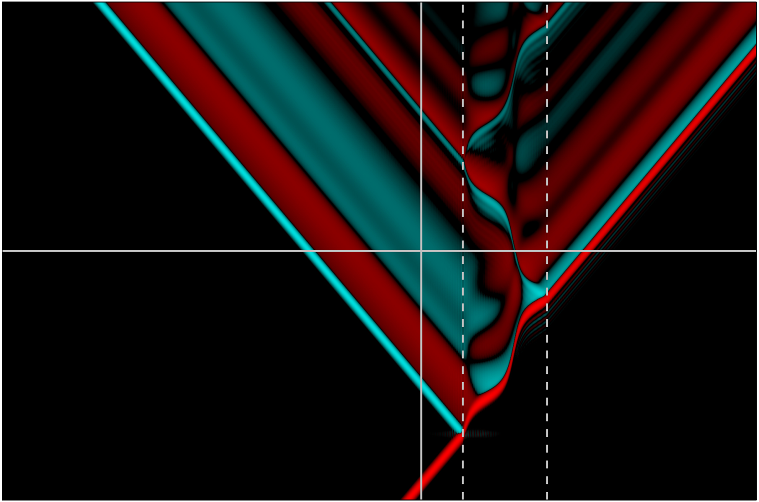

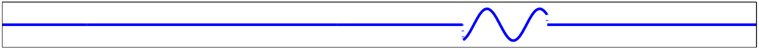

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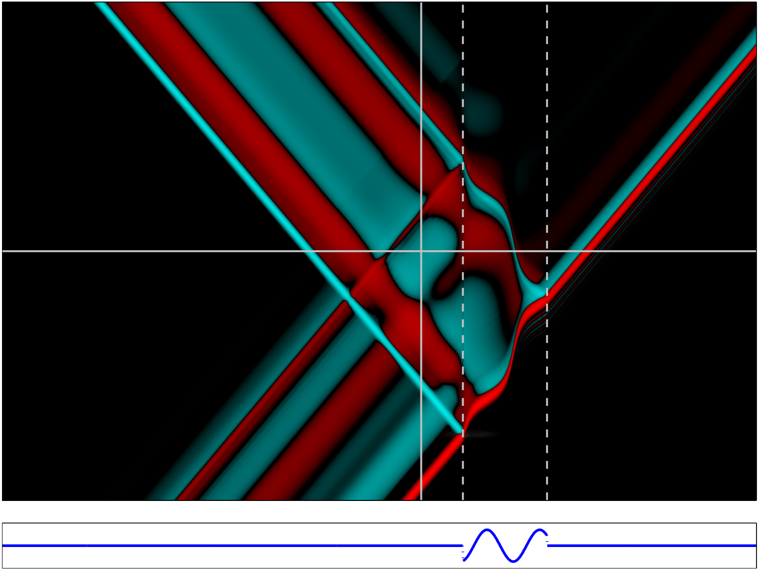

background

- Marchenko's classical integral equation solves the inverse scattering problem in dimension one
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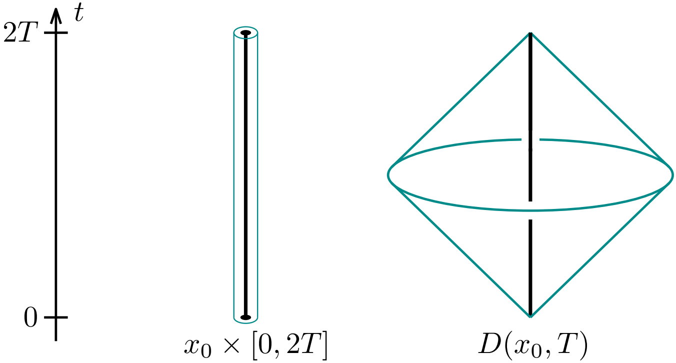
extend Marchenko approach to higher dimensional inverse problems (related work: Wapenaar, Thorbecke, Van der Neut, Broggini, Snieder, Curtis and others): key components are unique continuation (Tataru) and boundary control (Belishev)



a solution of the wave equation that is zero on the neighborhood on the left must be zero on the 'light diamond' assumptions

- $\Omega \subseteq \mathbb{R}^n$ (part of Earth's interior) is a Lipschitz domain
- c is a scalar wave speed:
  - unknown and piecewise smooth on Ω
  - known and smooth on $\Omega^\star = \mathbb{R}^n \setminus \overline{\Omega}$

initial value problem (IVP) and data model

let $h = (h_0, h_1) \in H^1(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n)$; consider the IVP

$$F\colon h \mapsto u \text{ s.t. } \begin{cases} \partial_t^2 u - c^2 \Delta u = 0 & \text{in } \mathbb{R} \times \mathbb{R}^n \\ u(0,\cdot) = h_0 & \text{in } \mathbb{R}^n \\ \partial_t u(0,\cdot) = h_1 & \text{in } \mathbb{R}^n \end{cases}$$

response after time s

$$\begin{aligned} R_s\colon H^1(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n) &\to H^1(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n) \\ h &\mapsto \left(Fh, \partial_t Fh\right)\Big|_{t=s} \end{aligned}$$

known: $R_{2T}h\big|_{x\in\Omega^\star}$ for Cauchy data h supported in $\Omega^\star$,

$\boxed{T \in (0, \frac{1}{2}\operatorname{diam}\Theta)}$

data operator: $\mathcal{F}\colon\; H^1(\Omega^\star) \oplus L^2(\Omega^\star) \to C(\mathbb{R}, H^1(\Omega^\star))$

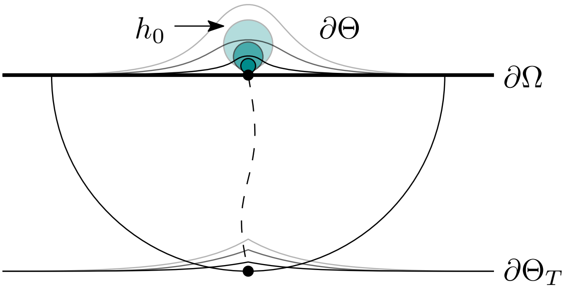
sets

- let $T > 0$, choose Lipschitz Θ, Υ s.t.

$$\overline{\Omega} \subset \Theta \subset \overline{\Theta} \subset \Upsilon$$

(think of $\Theta \approx \Omega$, and Υ a large ambient space)

sets, signed distance



Cauchy data, spaces

sublevel sets (d^*_Θ: signed distance to the boundary $\partial\Theta$)

$$\begin{aligned} \Theta_t &= \{x \in \Upsilon \mid d^*_\Theta(x) > t\} \\ \Theta^\star_t &= \{x \in \Upsilon \mid d^*_\Theta(x) < t\} \end{aligned}$$

(sub)spaces of Cauchy data

$$\widetilde{\mathbf{C}} = H^1_0(\Upsilon) \oplus L^2(\Upsilon)$$

$$\begin{aligned} \mathbf{H}_t &= H^1_0(\Theta_t) \oplus L^2(\Theta_t), \qquad \mathbf{H} = \mathbf{H}_0 \\ \widetilde{\mathbf{H}}^\star_t &= H^1_0(\Theta^\star_t) \oplus L^2(\Theta^\star_t) \end{aligned}$$

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$\widetilde{\mathbf{H}}^\star \cap (R_{2T}(H^1_0(\mathbb{R}^n \setminus \overline{\Theta}) \oplus L^2(\mathbb{R}^n \setminus \overline{\Theta})))$: space of Cauchy data in $\widetilde{\mathbf{C}}$ whose wave fields vanish on Θ at $t = 0$ and $t = 2T$; $R_{2T} :\ \mathbf{C} \to \mathbf{C}$ isometrically $\qquad \pi_{\mathbf{C}} :\ \widetilde{\mathbf{C}} \to \mathbf{C}$

$\mathbf{C}$: its orthogonal complement inside $\widetilde{\mathbf{C}}$

$\mathbf{H}^\star_t$: its orthogonal complement inside $\widetilde{\mathbf{H}}^\star_t$

norms and inner product

inner product on **C**

$$\big\langle (f_0, f_1), (g_0, g_1) \big\rangle = \int_{\Upsilon} \big(\nabla f_0(x) \cdot \nabla \overline{g}_0(x) + c^{-2} f_1(x) \overline{g}_1(x) \big) \; dx$$

energy in open set $W \subseteq \mathbb{R}^n$

$$\mathbf{E}_W(h) = \int_W \big(|\nabla h_0|^2 + c^{-2} |h_1|^2 \big) \; dx$$

kinetic energy

$$\mathbf{KE}_W(h) = \int_W c^{-2} |h_1|^2 \, dx$$

projections inside and outside Θ_t

- orthogonal projections

$$\pi_t : \ \mathbf{C} \to \mathbf{H}_t, \qquad \pi = \pi_0$$

$$\pi_t^\star : \ \mathbf{C} \to \mathbf{H}_t^\star, \qquad \pi^\star = \pi_0^\star$$

- $$\overline{\pi}_t = I - \pi_t^\star, \quad (\overline{\pi}_t h)(x) = \begin{cases} h(x), & x \in \Theta_t \\ (\phi(x), 0), & x \in \Theta_t^\star \end{cases}$$

  where ϕ is the harmonic extension of $h|_{\partial\Theta_t}$ to Υ (with zero trace on $\partial\Upsilon$)

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- set $R = \upsilon \circ R_{2T}$, where $\upsilon\colon (h_0, h_1) \mapsto (h_0, -h_1)$

  $\pi^\star R$: *reflection response operator*

scattering control

$h_0 := (h_{0,0}, h_{0,1})$

scattering control series: $T \in (0, \frac{1}{2}\,\mathrm{diam}\,\Theta)$

given Cauchy data h_0 supported in $\Theta \setminus \overline{\Omega}$, define Neumann series

$$h_\infty = (I - \pi^\star R \pi^\star R)^{-1} h_0 = \sum_{i=0}^{\infty} (\pi^\star R \pi^\star R)^i h_0$$

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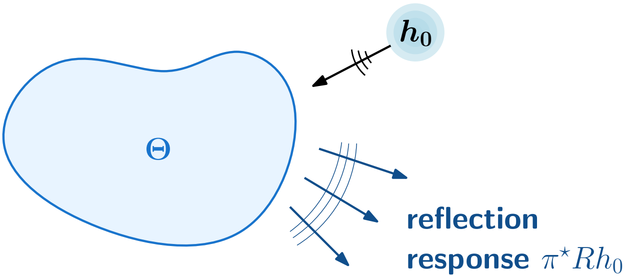

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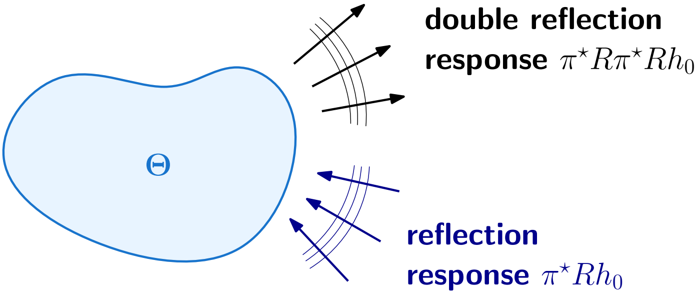

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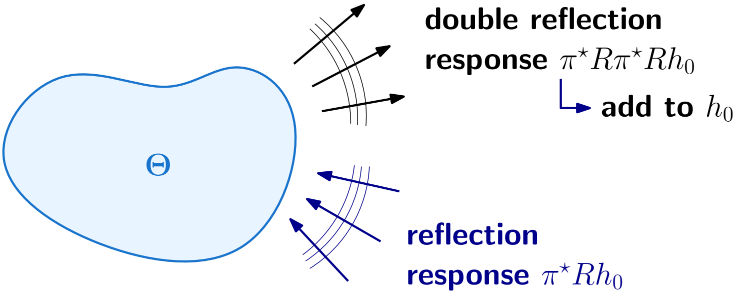
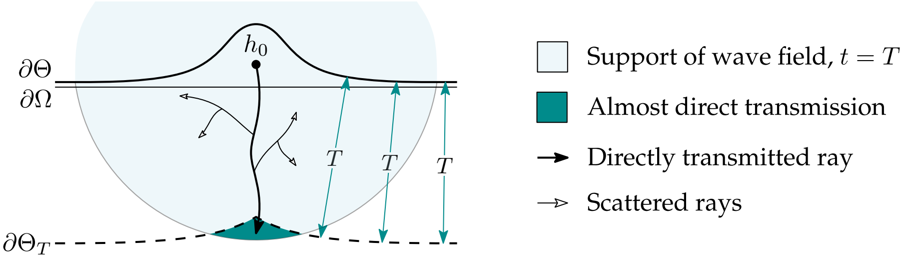


definition

- the *T-sublevel set* is $\Theta_T = \{x \in \Theta \mid d^*_\Theta(x) > T\}$
- the *almost direct transmission* of h_0 is $R_T h_0\big|_{\Theta_T}$
- the *harmonic almost direct transmission* is its harmonic extension, $h_{\mathrm{DT}} = \overline{\pi}_T R_T h_0$



series behavior (I)

let $h_0 \in \mathbf{H}$ *and* $T \in (0, \frac{1}{2}\operatorname{diam}\Theta)$*; finding the wave field of the harmonic almost direct transmission of* h_0 *is equivalent to summing the scattering control series:*

$$(I - \pi^\star R \pi^\star R) h_\infty = h_0 \iff \begin{array}{c} R_{-T}\overline{\pi} R_{2T} h_\infty = h_{DT} \\ \textit{and} \\ h_\infty - h_0 \in \mathbf{H}^\star \end{array}$$

such an h_∞*, if it exists, is essentially unique*

by unique continuation and finite speed of propagation; works for any c with these properties

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$$(I - \pi^\star R \pi^\star R) h_\infty = h_0 \iff \begin{array}{c} R_{-s}\overline{\pi}_{T-s} R_{T+s} h_\infty = h_{DT} \\ \text{and} \\ h_\infty - h_0 \in \mathbf{H}^\star, \end{array} \qquad s \in [0, T]$$

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series behavior (II)

let h_k be the Neumann series' k^{th} partial sum

$$h_k = \sum_{i=0}^{k} (\pi^\star R \pi^\star R)^i h_0$$

- *the wave field that h_{DT} generates can be recovered from $\{h_k\}$ regardless of convergence of the scattering control series:*

$$\lim_{k\to\infty} R_{-T} \overline{\pi} R_{2T} h_k = R_T \chi h_0 = h_{DT}$$

- *$\{h_k\}$ converges in energy space on a dense set*
- *$\{h_k\}$ always converges in a larger weighted space (spectral theorem)*

energy recovery (I)

- energy conservation allows us to find the energy of the almost direct transmission using only outside-observable data

the energy of the harmonic almost direct transmission (including harmonic extension) is

$$\mathbf{E}(h_{\mathrm{DT}}) = \mathbf{E}\big(h_\infty\big) - \mathbf{E}\big(\pi^\star R h_\infty\big)$$

the kinetic energy (not including harmonic extension) is

$$\mathbf{KE}(h_{\mathrm{DT}}) = \frac{1}{2}\langle h_0,\ h_0 - R\pi^\star R h_\infty - R h_\infty\rangle$$

energy recovery (II)

- even without convergence, one can recover the same energies as monotone limits

the energy of the harmonic almost direct transmission (including harmonic extension) is

$$\mathbf{E}(h_{\mathrm{DT}}) = \lim_{k\to\infty} \left[\mathbf{E}(h_k) - \mathbf{E}(\pi^\star R h_k)\right]$$

the kinetic energy (not including harmonic extension) is

$$\mathbf{KE}(h_{\mathrm{DT}}) = \frac{1}{4} \lim_{k\to\infty} \Big[\mathbf{E}(h_k) + \mathbf{E}(h_0) - \mathbf{E}(\pi^\star R \pi^\star R h_k) + 2\langle \pi^\star R h_k,\ h_k - R\pi^\star R h_k\rangle - 2\langle h_0,\ R\pi^\star R h_k + R h_k\rangle\Big]$$

broken boundary normal ('time') coordinates

- set of disjoint, closed, connected, smooth hypersurfaces: $\Gamma = \bigcup \Gamma_i$
- $\{\Omega_j\}$: the connected components of $\mathbb{R}^n \setminus \Gamma$

- we call $x \in \Omega$ *regular* if $x \notin \Gamma$ and the infimum in $d(x, \partial\Omega) = d(\{x\}, \partial\Omega)$ is achieved by a unique purely transmitted broken path that is nowhere tangent to Γ

Assumption A: almost every $x \in \Omega$ is regular

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suppose Ω is compact and the interfaces Γ_i are strictly convex, viewed from their interiors Ω_i, then the set of regular points, Ω_r, is open and dense in Ω

recovery of transformation of coordinates

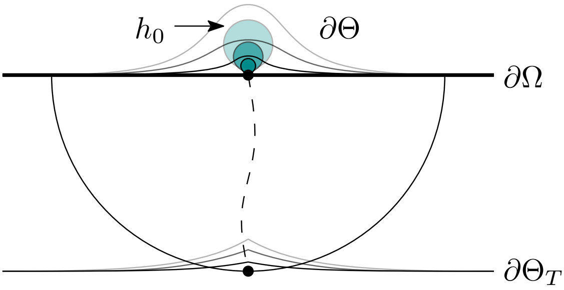
for any $h_0 \in \mathbf{C}$, f, g *harmonic*

$$\langle \overline{\pi}_T R_T h_0, (f, g) \rangle = \lim_{k \to \infty} \left[\langle h_k, (f - Tg, g) \rangle - \langle \pi^\star R_{2T} h_k, (f + Tg, g) \rangle \right]$$

if the scattering control series converges, h_k *can be replaced above by* h_∞ *and the limit omitted*

- the appeal of this result is that the harmonic almost direct transmission $\overline{\pi}_T R_T h_0$ may be arbitrarily spatially concentrated (aside from harmonic extensions in the first component)
- taking inner products with the harmonic data $(0, x^i)$ and $(0, 1)$, we may now recover weighted averages of x^i over this support

direct transmission, limit



recovery of transformation of coordinates

let $y = (y^1, \dots, y^n) \in \Omega_r$, $p = p(y) \in \partial\Omega$, *and* $T = d(y, \partial\Omega)$; *let* x^i *denote the* i*th Euclidean coordinate function*

choose a nested sequence of Lipschitz domains $\Theta^{(1)} \supset \Theta^{(2)} \supset \cdots \supset \Omega$ *such that* $\bigcap_j \Theta^{(j)} = \Omega \cup \{p\}$ *and* $\operatorname{diam} \Theta^{(j)} \setminus \Omega \to 0$; *then*

$$y^i = \Phi^i(p, T) = \lim_{j\to\infty} \frac{\kappa(\mathbf{1}_{\Theta^{(j)}\setminus\Omega}, x^i)}{\kappa(\mathbf{1}_{\Theta^{(j)}\setminus\Omega}, 1)}$$

where

$$\kappa(g, f) = \langle \bar{\pi}_T R_T(0, \pi_{\mathbf{C}} g), \, (0, f) \rangle$$

moreover

$$c = \left| \frac{\partial \Phi}{\partial T} \right|$$

wave speed – uniqueness

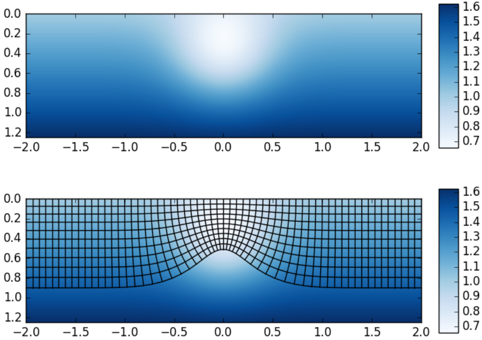
then c *is uniquely determined on* $\Omega^\star_T$ *by* $R_{2T}\big|_{\Omega^\star}$

proof

- completely constructive
- makes use of behavior of solutions near the boundary of their domains of influence
- uses the piecewise smooth structure to get information about the behavior of a progressive wave solution near the wavefront (avoiding times when the wavefront is tangent to an interface); to obtain the Euclidean coordinates of the interfaces, find the singularities of c after reconstruction

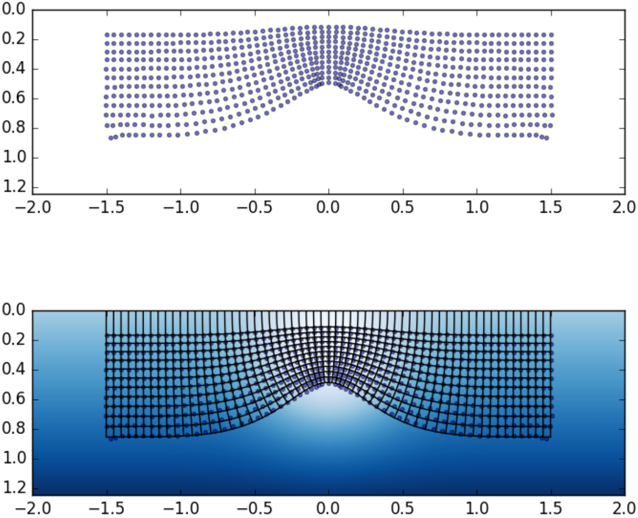
computational experiment – smooth wave speed

- $T = 1.0$
- $\Gamma = [-3.0, 3.0] \times \{0\}$
- $\mathcal{R} = [-4.5, 4.5] \times \{0\}$

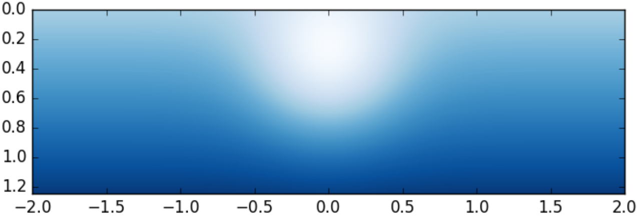


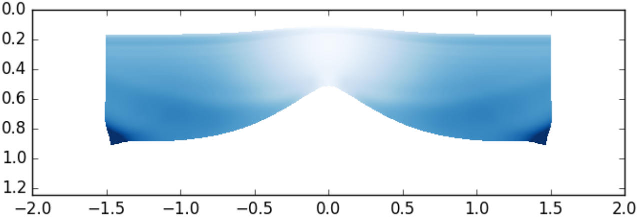
$$c(x_1, x_2) = 1 + \frac{1}{2}x_2 - \frac{1}{2}\exp\left(-4\left(x_1^2 + (x_2 - 0.375)^2\right)\right)$$

coordinate reconstruction from the data



wave speed reconstruction (dH, Kepley & Oksanen, '17)





detection of interfaces – direct broken transmission

transformation to half wave equations

$$\Lambda^{-1} = \frac{1}{2}\begin{bmatrix} I & iB^{-1} \\ I & -iB^{-1} \end{bmatrix}, \quad B^2 = -c^2\Delta$$

- principal symbol (amplitude) of the directly transmitted component $\mathbf{DT}^+$ of R^+ at $(p(y), \nu)$, where ν is the inward-pointing normal covector at p: $\mathbf{dt}^+(y)$
- wave packet of 'frequency' λ centered at (x, ξ): $\rho_{\lambda,x,\xi}\phi_{\lambda,x,\xi}$

strategy: *send in a wave packet, vary T, and track ADT energy*

- energy lost at each interface (discontinuity) to reflection
- drop sharper as frequency (λ) increases
- recover depths of interfaces in broken boundary normal coordinates through scattering control

detection of interfaces

let $y \in \Omega_r,\ p = p(y),\ T = d(y,p),\ \varepsilon > 0$ *be sufficiently small; then there exists a domain* $\Theta \supset \Omega$ *and covector* $(p^*, \nu^*) \in S^*\Theta$ *such that*

$$\left|\mathbf{dt}^+(y)\right|^2 = \lim_{\lambda\to\infty} \mathbf{KE}_{\Theta_{T+\varepsilon}} R_{T+\varepsilon} \Lambda \begin{bmatrix} -icB^{-1}\rho_{\lambda,p^*,\nu^*}\phi_{\lambda,p^*,\nu^*} \\ 0 \end{bmatrix}$$

because $\mathbf{dt}^+(y)$ is constant along a geodesic except at a discontinuity in c, we can recover the discontinuities of c in boundary normal coordinates:

if γ_y *is the broken geodesic connecting* y *to the surface,*

$$\gamma_y^{-1}(\Gamma) = \text{sing supp}(\left|\mathbf{dt}^+ \circ \gamma_y\right|)$$

movies: imaging interfaces without the wave speed

summary

'many' experiments (data),
piecewise smooth wave speed – completely unknown interfaces

- instantaneous time mirrors in extended subsurface
- scattering control and detection of kinetic energy by 'data manipulations'
- imaging interfaces without the wave speed
- coordinate transformation reconstruction $\rightarrow$ wave speed, discontinuities

◦ convex foliation condition implies stability

microlocal analysis – setup

microlocal adaptations

projections $\overline{\pi}$, $\pi^\star$	$\rightarrow$	smooth cutoffs σ, $\sigma^\star$ (supp $\sigma = \Theta$)
exact propagator R	$\rightarrow$	FIO parametrix $\widetilde{R}$

wave speeds

- singsupp $c = \Gamma = \cup_i \Gamma_i$
- Γ_i closed, connected, disjoint, smooth hypersurfaces in Θ
- $\widetilde{R}$ includes cutoffs removing glancing rays

microlocal scattering control equation

$$(I - \sigma^\star \widetilde{R} \sigma^\star \widetilde{R}) h_\infty \equiv h_0$$

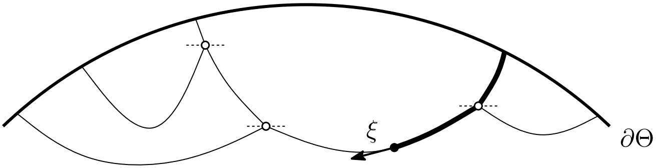
microlocal scattering control

microlocal scattering control equation

$$(I - \sigma^\star \widetilde{R} \sigma^\star \widetilde{R}) h_\infty \equiv h_0$$

if it exists, the tail $h_\infty - h_0$ still "erases the history" of h_0's wave field up to singularities at depth T; unlike exact analysis, depth is measured in $T^*\Theta$

the *depth* $d^*_{T^*\Theta}$ of a covector $\xi \in T^*\mathbb{R}^n \setminus 0$ is the length of the shortest broken bicharacteristic segment connecting it to $\partial T^*\Theta$:



microlocal "distance"

the *distance* of a covector $\xi \in T^*(\mathbb{R}^n \setminus \Gamma)$ from the boundary of $M \subseteq \mathbb{R}^n$ is

$$d(\xi, \partial T^*M) = \min\{|a-b| \mid \gamma(a) = \xi,\ \gamma(b) \in \partial T^*M\}$$

minimum taken over broken bicharacteristics γ (lack of continuity)

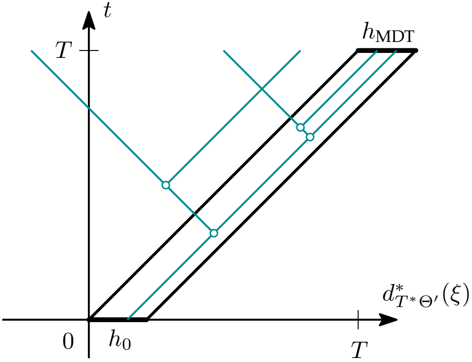
depth is the same as distance, but with a sign indicating whether ξ is inside or outside M

$$d^*_{T^*M}(\xi) = \begin{cases} +d(\xi, \partial T^*M), & \xi \in T^*M \\ -d(\xi, \partial T^*M), & \text{otherwise} \end{cases}$$

microlocal almost direct transmission – definition

the *T-sublevel set* is $(T^*\Theta)_T = \{\xi \in T^*\Theta \setminus 0 \mid d^*_{T^*\Theta}(\xi) > T\}$

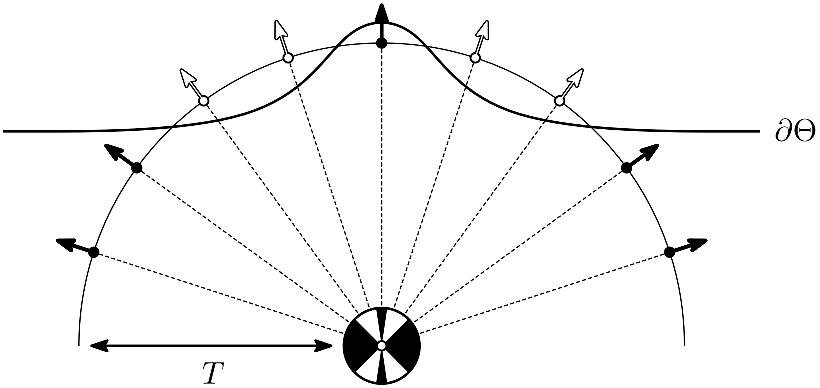
the *microlocal almost direct transmission* h_{MDT} of h_0 is a microlocal restriction of $R_T h_0$ to a neighborhood of $(T^*\Theta')_T$, $\Omega \subset \overline{\Theta'} \subset \Theta$



suppose $R_{2T} h_\infty|_\Theta = R_T h_{\mathrm{MDT}}|_\Theta$ then h_∞ satisfies microlocal scattering control equation

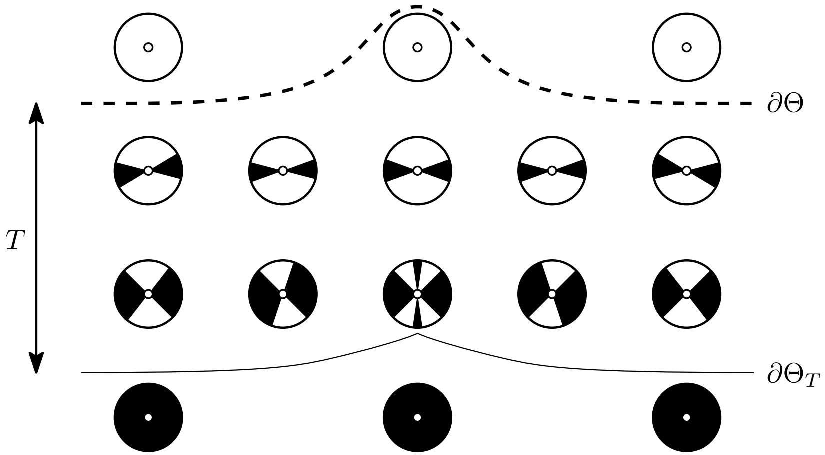
microlocal T-sublevel set

one fiber from $(T^*\Theta)_T$



microlocal T-sublevel set

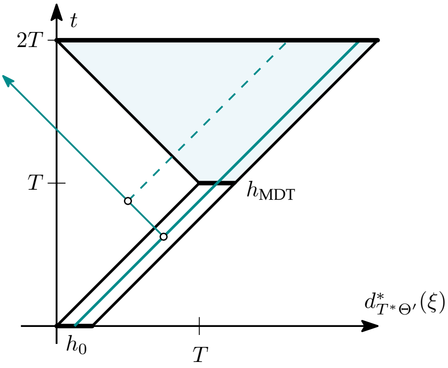
schematic illustration of $(T^*\Theta)_T$



constructive parametrix for $I - \sigma^\star \widetilde{R} \sigma^\star \widetilde{R}$

if c were known, can construct a microlocal inverse A for $I - \sigma^\star \widetilde{R} \sigma^\star \widetilde{R}$ valid for $\mathrm{WF}(h_0)$ in some conic $\mathcal{S} \subseteq T^*\Theta' \setminus 0$

A works by constructing appropriate singularities in the tail $h_\infty - h_0$ to prevent outside singularities from entering the domain of influence of h_{MDT}:



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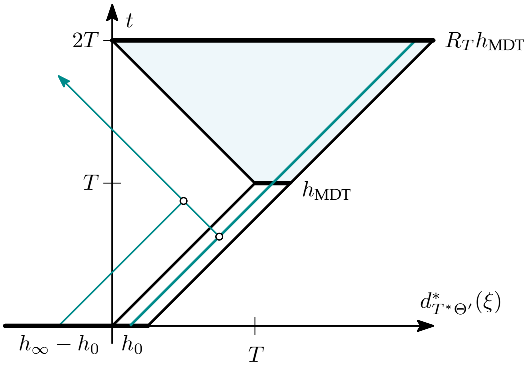


constructive parametrix for $I - \sigma^\star \widetilde{R} \sigma^\star \widetilde{R}$

define $(\pm)$-escapability through mutual recursion:

$$\gamma\colon (t_-, t_+) \to T^*(\mathbb{R}^n \setminus \Gamma)$$

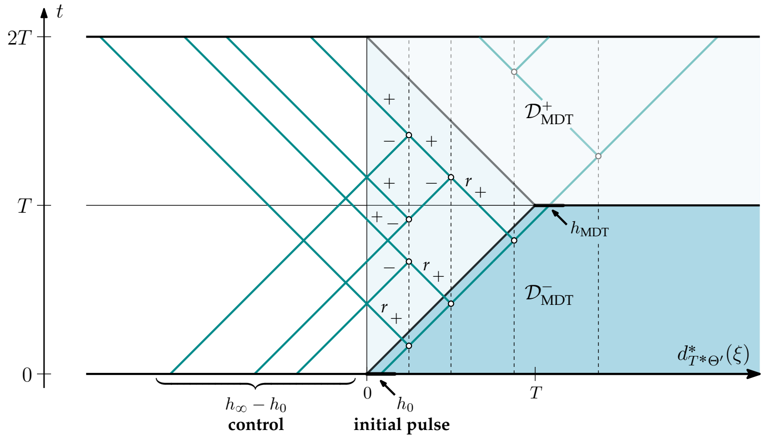
one of the following holds

- all of its connecting bicharacterisrics at $t_\pm$ are $(\pm)$-escapable
- one of its connecting bicharacterisrics at $t_\pm$ is $(\pm)$-escapable, the opposing bicharacteristic is $(\mp)$-escapable; if this $(\pm)$-escapable connecting bicharacteristic is a reflection, c must be discontinuous at $\gamma(t_\pm)$ (non-vanishing principal symbol)
- (base case) γ is defined at $t = T \pm T$ and $\gamma(T \pm T) \notin T^*\Theta$; γ reaches $t = 0$ ($-$ case) or $t = 2T$ ($+$ case) at a point outside Θ

may produce a singularity along a $(-)$-escapable bicharacteristic

$\mathcal{S} \subset T^*\Theta'$: the set of ξ so that every returning bicharacteristic γ with $\gamma(0) = \xi$ is $(+)$-escapable

returning and (±)-escapable bicharacteristics



convergence of Neumann iteration

Neumann iteration

let N_k be the Neumann series partial sum operators (order-0 FIOs)

$$N_k = \sum_{i=0}^{k} (\sigma^\star \widetilde{R} \sigma^\star \widetilde{R})^i$$

in general $\lim_{k\to\infty} N_k$ has no meaning $\Rightarrow$ consider principal symbol convergence

principal symbols

- standard microlocal splitting into $\pm$: split initial data $\mathbb{R}^n \to \mathbf{R}^n = \mathbb{R}^n \sqcup \mathbb{R}^n$, etc
- define principal symbols σ_0 of each graph component of the Lagrangian (polar decomposition)
  (each sequence of reflections, transmissions, and time-reversal)
- restrict to fiber $\mathcal{G}_\eta$: all covectors reachable from η with knowledge of the paths; principal symbols on $\mathcal{G}_\eta$

convergence of Neumann iteration

Theorem

on $\mathcal{S}$ the principal symbols of the N_k converge in $\ell^2(\mathcal{G}_\eta)$ to some n_∞; furthermore $\sigma_0(\widetilde{R}N_k) \to \sigma_0(\widetilde{R}A)$ in $\ell^2(\mathcal{G}_\eta \cap S^*\mathbf{\Theta}')$

- the space $\ell^2(\mathcal{G}_\eta)$ is microlocal analogue of bounded operators in energy space
- A reveals h_{MDT} in the sense that $\widetilde{R}_{2T}Ah_0 \equiv \widetilde{R}_T h_{\mathrm{MDT}}$ in Θ'
- convergence of principal symbols implies N_k “reveals h_{MDT} in the limit” when possible

convergence of Neumann iteration

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proof: show composition with $\sigma^\star \widetilde{R}$ is an $\ell^2(\mathcal{G}_\eta)$-bounded operator of norm ≤ 1 (microlocal energy conservation); analyze convergence with spectral theorem

boundary rigidity – $\dim \Omega \geq 3$, convex foliation

we can use geometric results on boundary and lens rigidity due to Stefanov, Uhlmann, and Vasy to recover the smooth parts of c – this also gives stability

we need an extension of their convex foliation condition to our piecewise smooth setting

Assumption B: $\rho : \overline{\Omega} \to [0, T]$ is a (piecewise) *convex foliation* for (Ω, c),

- $\partial\Omega = \rho^{-1}(0)$ and $\rho^{-1}(T)$ has measure zero
- each level set $\rho^{-1}(t)$ is strictly convex when viewed from $\rho^{-1}((t, T))$, for $t \in [0, T)$
- the interfaces of c are level sets of ρ: $\Gamma_i = \rho^{-1}(t_i)$ for some t_i
- ρ is smooth and $d\rho \neq 0$ on $\rho^{-1}((0, T)) \setminus \Gamma$

boundary rigidity, proof by contradiction

singular part of the data, $\mathcal{F}$, determine c almost everywhere

on Ω there are two notions of depth: d^*, the Riemannian distance to the boundary, and ρ that is defined by the foliation

by Snell's law and uniqueness of geodesics for smooth metrics, for any $(x, \xi) \in T^*\Omega \setminus 0$ there is a unique maximal transmitted bicharacteristic $\gamma_{x,\xi}$ satisfying $\gamma_{x,\xi}(0) = (x, \xi)$

- suppose $c \neq \widetilde{c}$, let $a = c - \widetilde{c}$; consider $S = \Omega_r \cap \operatorname{supp} a$, and take $\tau = \min_S \rho$
- let $\Sigma_\tau = \rho^{-1}(\tau)$ be the corresponding level set; let $\Omega_\tau = \rho^{-1}((\tau, T])$ be the corresponding sublevel set: so $c = \widetilde{c}$ above Ω_τ, but by compactness there is a point $x \in \rho^{-1}(\tau) \cap S$

boundary rigidity – orientation of a covector

a *foliation downward* (resp. *upward*) covector (x, ξ) is one pointing in the direction of increasing (resp. decreasing) ρ

Definition

$$T^*_{\pm}\Omega = \{(x, \xi) \in T^*\Omega_\tau \mid \pm \langle \xi, d\rho \rangle > 0\}$$

upward-traveling geodesics are not trapped

layer stripping – interior lens relation

- let $(x,\xi) \in T^*_+\overline{\Omega} \setminus 0$, $\tau = \rho(x)$; if there exists a purely transmitted bicharacteristic γ with $\lim_{t\to 0^+} \gamma(t) = (x,\xi)$, we define the *travel time* $\ell(x,\xi)$ as the unique $\ell > 0$ for which $\gamma(\ell) \in T^*_-\Omega \cap T^*\Sigma_\tau$, the *(interior) lens relation*: $L(x,\xi) = \gamma(\ell)$
- on the interfaces $T^*\Omega\big|_\Gamma$, define L by continuity from below

Lemma

(a) *Let $(x,\xi) \in \partial T^*\Omega_\tau \cap T^*_+\Omega$. Suppose c, $\widetilde{c}$ are smooth near x, and there are smooth bicharacteristics from (x,ξ) to $\partial T^*\Omega_\tau \cap T^*_-\Omega$ with respect to both wave speeds. If $\mathcal{F} = \widetilde{\mathcal{F}}$ and $c = \widetilde{c}$ outside Ω_τ, then c and $\widetilde{c}$ have identical subsurface lens relations w.r.t. the leaf Σ_τ at (x,ξ).*

(b) *Assume the same conditions as (a), except that c, $\widetilde{c}$ are discontinuous on Σ_τ near x, and $c - \widetilde{c}$ vanishes on both sides of Σ_τ near x. Then the same statement holds.*

boundary rigidity, proof by contradiction

- $x \notin \Gamma$: we use the fact that c, $\widetilde{c}$ are equal above Ω_τ to show that they locally have the same lens relation on Σ_τ; the additional wrinkle is that we must ensure that $\widetilde{c}$ is also smooth near x, which is where scattering control enters
- $x \in \Gamma$: c, $\widetilde{c}$ must have the same jump in wave speed at x because we can measure the transmission coefficient; once the jump is known (using convexity of Γ), the lens relation on the other side of the interface can be locally determined
- apply the local boundary rigidity theorem