

# Stochastic equations generating continuous multiplicative cascades

 F. Schmitt<sup>1,a</sup> and D. Marsan<sup>2</sup>
<sup>1</sup> Vrije Universiteit Brussel, Dept. of Fluid Mechanics, Pleinlaan 2, 1050 Brussels, Belgium

<sup>2</sup> LGIT, Université de Savoie Campus Scientifique, 73376 Le Bourget du Lac, Cedex, France

Received 26 October 2000 and Received in final form 2nd February 2001

**Abstract.** Discrete multiplicative turbulent cascades are described using a formalism involving infinitely divisible random measures. This permits to consider the continuous limit of a cascade developed on a continuum of scales, and to provide the stochastic equations defining such processes, involving infinitely divisible stochastic integrals. Causal evolution laws are also given. This gives the first general stochastic equations which generate continuous multifractal measures or processes.

**PACS.** 02.50.Ey Stochastic processes – 05.40.Fb Random walks and Lévy flights

## 1 Introduction

Multiplicative cascades were first introduced in turbulence to model the energy flux in the inertial range. The cascade formalism was originally introduced as a discrete (in scale) procedure, with a fixed (often 2) scale ratio between the scale of a structure and that of the daughter structures (see [1–4]). Discrete cascade models lead to discrete scale invariance, characterized by log-periodic modulations [5–8]. On the other hand, a continuous symmetry leading to scaling for any scale ratio has been proposed, and corresponds to a scale densification of cascade models [9–14]. Scale densification implies the use of infinitely divisible (ID) random variables, defining cascade models that can be called log-ID [10,11,13], and have been compared to experimental data in various studies [15–20]. This then leads to an interrogation: discrete cascade models are built using a simple recursive multiplicative procedure (see below), but what is the continuous limit of this procedure? What is the stochastic equation generating continuous multifractals? Up to now, the process generated by such continuous multiplicative cascades has not been explicitly described in the general case; only its statistical moments are given [13], or some general relation verified by the pdf at different scales [21]. This paper aims at detailing how ID random measures can be introduced for discrete cascades; their continuous limit is then given in the form of ID stochastic integrals. The stochastic evolution laws that generate causal continuous multifractal processes will also be provided. This is of direct importance for providing estimators of the future state of the process.

## 2 Discrete multiplicative cascades

Soon after Kolmogorov and Obukhov published their log-normal proposal for the statistics of the small-scale dissipation field [22,23], experimental studies showed that the dissipation field had long-range power-law correlations [24,25]. This led Yaglom to propose a random cascade model with long-range correlations and small-scale lognormal statistics [1]. Yaglom’s multiplicative cascade model is at the basis of most cascade models introduced later to account for turbulent intermittency. It is a discrete (in scale) model, but most of its properties are shared by continuous models. A lognormal pdf is assumed, but this is an unnecessary hypothesis, as is now well recognised. This model is multiplicative, nested in a recursive manner. The multiplicative hypothesis generates large fluctuations, and the stacking generates long-range correlations, giving spatially to these large fluctuations their intermittent character.

As is classically done (see *e.g.* Frisch [26]), we define the cascade yielding a dissipation field  $\epsilon(x)$  at the smallest scale  $\ell_0$ , as the product

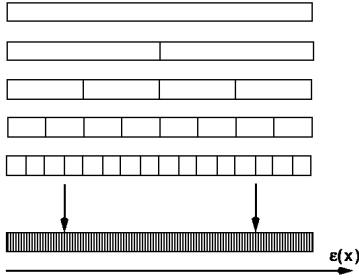
$$\epsilon(x) = \prod_{i=1}^n W_{i,x} \quad (1)$$

of  $n$  independent realisations  $W_{i,x}$  of a common, positive law (see Fig. 1). The cascade is developed from the largest scale  $L$  down to  $\ell_0 = L/\Lambda$  where  $\Lambda = \lambda_1^n$  is the total scale ratio and  $\lambda_1 > 1$  is the constant scale ratio between two consecutive scales. Generally one assumes for convenience  $\lambda_1 = 2$ , but we will later consider the  $\lambda_1 \rightarrow 1$  limit corresponding to a continuous cascade [27]. Since all random variables are independent, one has the moments of order  $q > 0$  of  $\epsilon$ :

$$\langle (\epsilon(x))^q \rangle = \prod_{i=1}^n \langle (W_{i,x})^q \rangle = \langle W^q \rangle^n = \Lambda^{K(q)} \quad (2)$$

---

<sup>a</sup> e-mail: francois@stro.vub.ac.be



**Fig. 1.** Schematic representation of a discrete multiplicative cascade.

where  $K(q) = \log_{\lambda_1} \langle W^q \rangle$ . For discrete cascades, the analytical expression taken by  $K(q)$  is only loosely constrained *a priori*: by conservation  $K(0) = 0$ ,  $K(1) = 0$ , and since  $K(q)$  is – up to a  $\log \lambda_1$  factor – the second Laplace characteristic function of the random variable  $\log W$ , it is a convex function (see [28]).

To densify the cascade described above, we keep the total scale ratio  $\Lambda$  large but fixed; the continuous limit can be obtained by increasing the total step number  $n$ , hence  $\lambda_1 = \Lambda^{1/n} \rightarrow 1^+$  (see [13, 19, 27, 29]). Equation (1) then shows that, in this limit,  $\log \epsilon$  is an ID random variable (see [28] for ID random variables): continuous cascade models are log-ID [10, 11, 13, 14]. This restricts the eligible cascade models, since ID laws define a specific family of probability distributions.

Let us also mention one of the main properties of multiplicative cascades: long-range power-law correlations. Following the development given by Yaglom [1], one can consider two points separated by a distance  $r$  as having common ancestors from steps 1 to  $p$ , and separated (hence the corresponding random variables are independent) paths for steps  $p$  to  $n$ , where  $\lambda_1^p \approx r$ . Direct calculations then provides the classical result for two-points correlations of multifractal fields [30, 31]:

$$\langle \epsilon(x)^p \epsilon(x+r)^q \rangle \approx \Lambda^{K(p+q) - r^{K(p)+K(q)-K(p+q)}} \quad (3)$$

for  $p > 0$  and  $q > 0$ . Since  $K(q)$  is non-linear for multifractal distributions, the exponent  $K(p) + K(q) - K(p+q)$  quantifies the long-range power law correlations of multifractal measures. For  $p = q = 1$ , this yields the  $\mu = K(2)$  exponent originally given for usual correlations by Yaglom [1].

The scaling law for the moments (1) and the power-law correlations (3) are the two signatures of multifractality, that are to be recovered by multifractal stochastic models, as we define them below.

### 3 ID random measures and stochastic integrals

The densification of a multiplicative cascade implies that  $\log \epsilon$  is an ID random variable; we now express this densification in the form of ID stochastic integrals. Since we need below to consider moments of order  $q > 0$  of the ID random variable  $\Gamma = \log \epsilon$ , we consider ID laws for which the

second Laplace characteristic function  $\Psi_X(q) = \log \langle e^{qX} \rangle$  converges for a given domain  $\Theta$ . For an ID random variable  $X$ , one has the general result that  $\forall n$  integer,  $\frac{\Psi_X(q)}{n}$  is still a second characteristic function. This shows that a family of ID laws can be defined: two ID laws belong to the same family if their second characteristic function is proportional. Then each ID family can be characterized by a reference function  $\Psi_0(q)$ . We choose a reference function such that  $\Psi_0(1) = 1$ , and for an ID random variable  $X$ , we define its scale  $S(X)$  as the proportionality factor, giving the general identity  $\Psi_X(q) = S(X) \Psi_0(q)$ . The scale is a positive real number; it is an additive function since for two independent ID random variables  $X$  and  $Y$  of the same family, it is easily checked that we have  $S(X+Y) = S(X) + S(Y)$ . An ID random variable is then uniquely characterized by its reference second characteristic function and its scale (relative to this reference function). As examples,  $\Psi_0(q) = q^2$  for a Gaussian law, and  $\Psi_0(q) = \log_2(1+q)$  for a Gamma law.

We then define ID random measures as set functions  $M(A)$ , such that  $\forall A$ ,  $M(A)$  is an ID random variable, with a scale given by  $S(M(A)) = m(A)$ ,  $m(A)$  being the control measure of  $M(A)$ . We can easily check that  $M(A)$  possesses the basic additive property of random measures: for two sets  $A$  and  $B$  with  $A \cap B = \emptyset$ , let us note  $C = A \cup B$ . By definition,  $M(C)$  is a ID law, and its scale verifies:  $S(M(C)) = m(C) = m(A) + m(B) = S(M(A)) + S(M(B))$ , hence that  $M(C) = M(A) + M(B)$ . This corresponds to the following second characteristic function for  $M(A)$ :

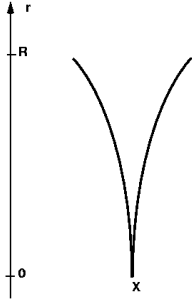
$$\Psi_{M(A)}(q) = m(A) \Psi_0(q). \quad (4)$$

This expression takes a more familiar form in the 1D case when  $A = [0, t]$  is an interval, and taking  $M(A) = Y(t) - Y(0) = Y(t)$  where  $Y$  is a process with independent and stationary increments. Then one has the classical result  $\Psi_{Y(t)}(q) = t \Psi_Z(q)$  where  $Z$  is the stationary process given by  $Z(t) = Y(t) - Y(t-1)$ . In the following we keep the random measure notation, which is more general.

Having introduced an ID random measure  $M$ , a stochastic integral can be built (see *e.g.* [33]), as a Stieltjes integral:

$$\int_a^b f(t) M(dt) = \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} f\left(a + i \frac{b-a}{n}\right) \times M\left(a + i \frac{b-a}{n}, a + (i+1) \frac{b-a}{n}\right). \quad (5)$$

As we will see below, the densification of the cascade leads to a stochastic integral with  $f(t) = 1 \forall t$ . In this case, the second characteristic function of the integral has a simple expression. Let us note  $I = \int_A M(dx)$ . By additive property,  $I$  is still an ID law of the same family as  $M$ , and its scale is given by  $S(I) = S(\int_A M(dx)) = \int_A dx = m(A)$ , such that we have still equation (4) with  $M(A) = \int_A M(dx)$  and  $m(A) = \int_A dx$ . Let us note that when  $f$  is not identically 1, the result is not so simple, since an addition of ID random variables belongs to the same family, but not a linear combination.



**Fig. 2.** Conic structure starting at position  $x$ , corresponding to the integration surface to obtain  $\Gamma(x)$ .

One has in this case (if the integral converges, see [33]):  $\Psi_I(q) = \int_A \Psi_0(qf(x)) dx$ , showing that in general,  $\Psi_I(q)$  is not proportional to  $\Psi_0(q)$ .

#### 4 Densification of the cascade and stochastic equations

Let us introduce  $\lambda$  a variable scale ratio, verifying  $1 \leq \lambda \leq \Lambda$ , where  $\Lambda$  is the fixed largest scale ratio. We also introduce  $R = \log \Lambda$  and  $r = \log \lambda$ . The elementary scale ratio introduced above writes now  $\lambda_1 = \lambda^{1/n} = e^{R/n}$ . The discrete cascade corresponds to introducing a stochastic kernel  $M$  and intervals  $A_p$  and  $B_p$  such that

$$\Gamma(x) = \log \epsilon(x) = \sum_{p=0}^{n-1} M(A_p, B_p(x)) \quad (6)$$

where here  $M(A, B)$  is a random variable depending only on  $m(A)$ , giving  $\Psi_{M(A,B)}(q) = m(A)\Psi_0(q)$ . The intervals  $A_p$  and  $B_p$ , responsible for the cascading parent/children structure, are built in the following way: the width of  $A_p$  is linear in  $r$ , giving  $A_p = [\frac{pR}{n}, \frac{(p+1)R}{n}]$ . The intervals  $B_p(x)$  are centered in  $x$  and of width proportional to  $\lambda_1^p = e^{pR/n}$ , giving  $B_p(x) = [x - \frac{\tau}{2}e^{pR/n}, x + \frac{\tau}{2}e^{pR/n}]$  where  $\tau = L/\Lambda$  is the resolution. The densification, corresponding to  $n \rightarrow \infty$ , transforms then equation (6) into a stochastic integral and using equation (5) (with  $f=1$ ), we obtain finally that:

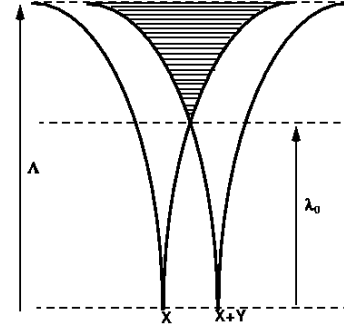
$$\epsilon_\Lambda(x) = \Lambda^{-c} \exp \int_1^\Lambda M\left(\frac{cd\lambda}{\lambda}, D_\lambda I_0(x)\right) \quad (7)$$

where  $c > 0$  is a parameter,  $I_0(x)$  is the interval of length  $\tau$  centered in  $x$ , and  $D_\lambda$  is the dilatation operator of factor  $\lambda$ . At a given position  $x$ , the stochastic integral corresponds to a kernel visiting a conical structure, as represented in Figure 2. This expression can be generalized to  $d$ -dimensional domains, and also to anisotropic scaling symmetries.

It can be easily verified that this stochastic equation generates a multifractal field. Indeed, the moments are scaling as  $\langle \epsilon_\Lambda^q \rangle = \Lambda^{K(q)}$  with

$$K(q) = c(\Psi_0(q) - q). \quad (8)$$

Moreover the two-points statistics can also be recovered: as was done above for equation (3), the correlation



**Fig. 3.** Intersection of two conic structures centered at positions  $x$  and  $x + y$ .

$\langle (\epsilon_\Lambda(x))^p (\epsilon_\Lambda(x+y))^q \rangle$  involves two integrals which have no intersection (and thus are independent random variables) for  $\lambda < \lambda_0$ , where  $\lambda_0 = e^{r_0} = \frac{\lambda}{\tau}$ .  $\lambda_0$  is the scale ratio of transition, as shown in Figure 3, and for  $\lambda_0 \leq \lambda \leq \Lambda$ , the random variables corresponding to the two stochastic integrals are no more independent. After some calculations, this leads to the same expression as equation (3) for discrete cascades.

We also give the expression for causal cascades, where the position is time and the past does not depend on the future. This case is of particular importance for prediction of multifractal time series. In this case, one can readily modify the intervals  $B_p(t)$  by taking an interval of the same length as before but preceding  $t$ :  $B_p(t) = [t - \tau \exp(pR/n), t]$ . This gives the following causal stochastic evolution law for continuous multifractals:

$$\epsilon_\Lambda(t) = \Lambda^{-c} \exp \int_1^\Lambda M\left(\frac{cd\lambda}{\lambda}, [t - \tau\lambda, t]\right) \quad (9)$$

Let us finally consider an important family, corresponding to logstable multifractals [9,32], including the lognormal case. Stable laws are ID, and possess a stronger property corresponding to stability, which can be written here  $M(kA) \doteq k^{1/\alpha} M(A)$  for  $k > 0$ , where ‘ $\doteq$ ’ means equality in distribution and  $0 \leq \alpha \leq 2$  is the Lévy index;  $\alpha = 2$  for the Gaussian case [33,34]. We have here  $\Psi_0(q) = q^\alpha$ ; we note that when  $\alpha < 2$ , the second Laplace characteristic function is defined for positive moments only for asymmetric laws for which hyperbolic pdf corresponds to negative fluctuations ( $\Pr(-X > x) \approx x^{-\alpha}$ ), whereas positive fluctuations have an exponential decay [9,32]. Then, by splitting equation (6) into two integrals, corresponding to backward and forward domains, and introducing the change of variables  $u = x - \frac{\tau}{2}\lambda$  and  $v = x + \frac{\tau}{2}\lambda$  respectively, one obtains (introducing the Lévy measure  $L_\alpha(du) = M(du, [u, x])$ ) a stable stochastic integral:

$$\epsilon_\Lambda(x) = \Lambda^{-c} \exp \int_{A(x)} |u - x|^{-1/\alpha} dL_\alpha(cu) \quad (10)$$

where  $A(x) = [x - X/2, x - \tau/2] \cup [x + \tau/2, x + X/2]$  and  $\Lambda = X/\tau$ . This equation corresponds to the exponential of a fractional integration (over a limited domain) of

order  $1 - 1/\alpha$  of a Lévy-stable noise. This expression was already given [9,31] by phenomenological arguments, and is here derived as a direct consequence of the densification. When the position is time, we obtain the following causal stochastic evolution equation for a logstable multifractal generated with a fixed scale ratio  $\Lambda = T/\tau$ :

$$\epsilon_{\Lambda}(t) = \Lambda^{-c} \exp \int_{t-T}^{t-\tau} (t-u)^{-1/\alpha} dL_{\alpha}(cu). \quad (11)$$

This can be directly used in numerical simulations. For lognormal multifractals  $L_{\alpha}$  is replaced by the Wiener measure  $W$ .

## 5 Conclusion

We now discuss the new results provided by our approach, compared to related papers. She and Waymire [13] have given a general expression for  $K(q)$  for continuous cascades, using the canonical Lévy-Khinchine representation for the second characteristic function of ID laws. This could of course also be provided here. On the other hand, we have pushed further the analysis, since the process itself was not studied in [13]. Castaing and collaborators have studied the convolution properties of the probability density of velocity increments under a change of scale, for continuous cascades [14,18,21]. This property is recovered here (it can be obtained from equation (7)), and we also provide the stochastic equation for the process itself. Other studies have provided Fokker-Planck [35] or Langevin [36] equations for the cascade process in the scale-ratio space. These equations apply only to lognormal cascades. These studies consider a fixed spatial (or temporal) position and provide stochastic equations for the cascade process developing at this particular position. This framework is then different from ours by the fact that (i) the position is not taken into account, whereas we included the position in our equations, and (ii) we provided equations for the general log-ID case, and not only for the lognormal case.

We have obtained continuous multifractals as the exponential of a stochastic integral. This formalization is of great interest for theoretical studies since it provides the first general stochastic equations for continuous multifractals. These equations can be generalized to anisotropic situations, and their dynamical properties (*i.e.* error growth and the corresponding predictability limit, or return times) are now open to theoretical studies, whereas before such studies were possible only through numerical simulations. We have also provided the general evolution equation for causal continuous multifractal processes. We have shown how these equations simplify for log-stable and lognormal multifractals.

These equations can be used for numerical simulations of continuous multifractals. In the general log-ID case, equation (7) can be used as follows: a 2D ID noise must be generated, in the  $r$  ( $r = \log \lambda$  is the logarithm of the scale ratio) -position plane. Then for each position, a numerical path-integration is performed in this plane, as given by this equation. In the stable case the procedure is simpler,

since it is enough to simulate a 1D stable (or Gaussian) noise and to proceed to a fractional integration over this noise, as given by equation (10).

The impetus to undertake this work has been given some time ago by C. Nicolis, which is gratefully acknowledged. The authors thank also S. Lovejoy, T. Over, D. Schertzer, S. Vannitsem for useful discussions. An anonymous referee is thanked for useful comments.

## References

1. A.M. Yaglom, Sov. Phys. Dokl. **11**, 26 (1966).
2. B.B. Mandelbrot, J. Fluid Mech. **62**, 305 (1974).
3. D. Schertzer, S. Lovejoy, in *Turbulence and Chaotic Phenomena in Fluids*, edited by T. Tatsumi (Elsevier, Amsterdam, 1984), p. 505.
4. R. Benzi, *et al.*, J. Phys. A **17**, 3521 (1984).
5. E. Novikov, Sov. Phys. Dokl. **11**, 497 (1966).
6. E. Novikov, Phys. Fluids A **2**, 814 (1990).
7. D. Sornette, Phys. Rep. **297**, 239 (1998).
8. D. Sornette, Fluid Mech. Appl. **46**, 251 (1998).
9. D. Schertzer, S. Lovejoy, J. Geophys. Res. **92**, 9693 (1987).
10. Y. Saito, J. Phys. Soc. Jpn **61**, 403 (1992).
11. E.A. Novikov, Phys. Rev. E **50**, R3303 (1994).
12. B. Dubrulle, Phys. Rev. Lett. **73**, 959 (1994).
13. Z.-S. She, E. Waymire, Phys. Rev. Lett. **74**, 262 (1995).
14. B. Castaing, J. Phys. II France **6**, 105 (1996).
15. F. Schmitt, *et al.*, Phys. Rev. Lett. **68**, 305 (1992).
16. Z.-S. She, E. Leveque, Phys. Rev. Lett. **72**, 336 (1994).
17. R. Benzi, *et al.*, Physica D **80**, 385 (1995).
18. F. Chilla, J. Peinke, B. Castaing, J. Phys. II France **6**, 455 (1996).
19. D. Schertzer, *et al.*, Fractals **5**, 427 (1997).
20. A. Arneodo, S. Manneville, J.-F. Muzy, Eur. Phys. J. B **1**, 129 (1998).
21. B. Castaing, Y. Gagne, E. Hopfinger, Physica D **46**, 177 (1990).
22. A.N. Kolmogorov, J. Fluid Mech. **13**, 82 (1962).
23. A.M. Obukhov, J. Fluid Mech. **13**, 77 (1962).
24. A.S. Gurvich, S.L. Zubkovski, Izv. Akad. Nauk. SSSR Ser. Geofiz. **12**, 1856 (1963).
25. S. Pond, R.W. Stewart, Izv. Akad. Nauk. SSSR Ser. Fiz. Atmosfer Okeana **1**, 914 (1965).
26. U. Frisch, *Turbulence; the legacy of A.N. Kolmogorov* (Cambridge University Press, Cambridge, 1995).
27. D. Schertzer, S. Lovejoy, in *Scaling, fractals and non-linear variability in geophysics*, edited by D. Schertzer, S. Lovejoy (Kluwer, Berlin, 1991), p. 41.
28. W. Feller, *An Introduction to Probability Theory and its Applications, Vol. II*, 2nd edn. (Wiley, New York, 1971).
29. F. Schmitt, S. Vannitsem, A. Barbosa, J. Geophys. Res. **103**, 23181 (1998).
30. M.E. Cates, J.M. Deutsch, Phys. Rev. A **35**, 4907 (1987).
31. D. Marsan, D. Schertzer, S. Lovejoy, J. Geophys. Res. **101**, 26333 (1996).
32. S. Kida, J. Phys. Soc. Jpn **60**, 5 (1991).
33. A. Janicki, A. Weron, *Simulation and chaotic behavior of  $\alpha$ -stable stochastic processes* (Marcel Dekker, New York, 1994), 355 pp.
34. G. Samorodnitsky, M.S. Taqqu, *Stable non-Gaussian random processes: stochastic models with infinite variance* (Chapman & Hall, New York, 1994), 632 pp.
35. R. Friedrich, J. Peinke, Phys. Rev. Lett. **78**, 863 (1997).
36. P. Marcq, A. Naert, Physica D **124**, 368 (1999).