Role of scattering in correlation-based imaging in random media

Josselin Garnier (Université Paris Diderot)
http://www.proba.jussieu.fr/~garnier/

with George Papanicolaou (Stanford University) and Chrysoula Tsogka (University of Crete).
Active imaging through a homogeneous medium

• Sensor array imaging of a reflector located at $\vec{y}$. $\vec{x}_s$ is a source, $\vec{x}_r$ is a receiver.

Data: \( \{ u(t, \vec{x}_r; \vec{x}_s), r = 1, \ldots, N_r, s = 1, \ldots, N_s \} \).

\[
\frac{1}{c_0^2} \left( 1 + \sigma_{\text{ref}} 1_{B_{\text{ref}}} (\vec{x} - \vec{y}) \right) \frac{\partial^2 u}{\partial t^2} (t, \vec{x}; \vec{x}_s) - \Delta_\vec{x} u(t, \vec{x}; \vec{x}_s) = f(t) \delta(\vec{x} - \vec{x}_s)
\]
Active imaging through a homogeneous medium

- Sensor array imaging of a reflector located at $\vec{y}$. $\vec{x}_s$ is a source, $\vec{x}_r$ is a receiver.

Data: \{\(u(t, \vec{x}_r; \vec{x}_s)\), \(r = 1, \ldots, N_r\), \(s = 1, \ldots, N_s\)\}.

\[
\frac{1}{c_0^2}(1 + \sigma_{\text{ref}}1_{B_{\text{ref}}}(\vec{x} - \vec{y})) \frac{\partial^2 u}{\partial t^2}(t, \vec{x}; \vec{x}_s) - \Delta \vec{x}u(t, \vec{x}; \vec{x}_s) = f(t)\delta(\vec{x} - \vec{x}_s)
\]

- Image with Kirchhoff Migration:

\[
\mathcal{I}_{\text{KM}}(\vec{y}^S) = \sum_{r=1}^{N_r} \sum_{s=1}^{N_s} u(\mathcal{T}(\vec{x}_s, \vec{y}^S) + \mathcal{T}(\vec{y}^S, \vec{x}_r), \vec{x}_r; \vec{x}_s)
\]

It forms the image with the superposition of the backpropagated traces. $\mathcal{T}(\vec{y}^S, \vec{x})$ is the travel time from $\vec{x}$ to $\vec{y}^S$, i.e. $\mathcal{T}(\vec{y}^S, \vec{x}) = |\vec{y}^S - \vec{x}|/c_0$. 

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Kirchhoff Migration:

\[
I_{\text{KM}}(\vec{y}^S) = \sum_{r=1}^{N_r} \sum_{s=1}^{N_s} u(T(\vec{x}_s, \vec{y}^S) + T(\vec{y}^S, \vec{x}_r), \vec{x}_r; \vec{x}_s)
\]

\[
I_{\text{KM}}(\vec{y}^S) = \frac{1}{2\pi} \sum_{r=1}^{N_r} \sum_{s=1}^{N_s} \int \hat{u}(\omega, \vec{x}_r; \vec{x}_s) \exp \left\{ i\omega [T(\vec{x}_s, \vec{y}^S) + T(\vec{y}^S, \vec{x}_r)] \right\} d\omega
\]

- When \( \lambda \ll a \ll L \):
  Cross-range resolution: \( \lambda L/a \), where \( \lambda \) is the central wavelength, \( L \) is the distance from the array to the reflector, and \( a \) is the array diameter.
  Range resolution: \( c_0/B \), where \( c_0 \) is the background velocity and \( B \) is the bandwidth.

- Very robust with respect to additive measurement noise.

- Sensitive to clutter noise (scattering medium): If the medium is scattering, then Kirchhoff Migration (usually) does not work.
Imaging through a weakly scattering medium

Sensor array imaging of a reflector located at $\vec{y}$. $\vec{x}_s$ is a source, $\vec{x}_r$ is a receiver.

Data: $\{u(t, \vec{x}_r; \vec{x}_s), r = 1, \ldots, N_r, s = 1, \ldots, N_s\}$.

If the medium is weakly scattering, then Kirchhoff migration does not work:

$$|I_{KM}(\vec{y}^S)|^2 = \left| \sum_{s=1}^{N_s} \sum_{r=1}^{N_r} \int \hat{u}(\omega, \vec{x}_r; \vec{x}_s) \exp \left\{ i\omega \left[ T(\vec{x}_s, \vec{y}^S) + T(\vec{y}^S, \vec{x}_r) \right] \right\} d\omega \right|^2$$

$$= \sum_{s,s'=1}^{N_s} \sum_{r,r'=1}^{N_r} \int \int d\omega d\omega' \hat{u}(\omega, \vec{x}_r; \vec{x}_s) \hat{u}(\omega', \vec{x}_{r'}; \vec{x}_{s'})$$

$$\times \exp \left\{ -i\omega \left[ T(\vec{x}_r, \vec{y}^S) + T(\vec{y}^S, \vec{x}_s) \right] + i\omega' \left[ T(\vec{x}_{r'}, \vec{y}^S) + T(\vec{x}_{s'}, \vec{y}^S) \right] \right\}$$

Problem because $\hat{u}(\omega, \vec{x}_r; \vec{x}_s)$ and $\hat{u}(\omega', \vec{x}_{r'}; \vec{x}_{s'})$ can be uncorrelated.
Sensor array imaging of a reflector located at $\vec{y}$. $\vec{x}_s$ is a source, $\vec{x}_r$ is a receiver.

Data: $\{u(t, \vec{x}_r; \vec{x}_s), r = 1, \ldots, N_r, s = 1, \ldots, N_s\}$.

If the medium is weakly scattering, then image with Coherent Interferometric Imaging (CINT) [1]:

$$I_{\text{CINT}}(\vec{y}^S) = \sum_{s,s'=1}^{N_s} \sum_{r,r'=1}^{N_r} \int \int d\omega d\omega' \hat{u}(\omega, \vec{x}_r; \vec{x}_s) \overline{\hat{u}(\omega', \vec{x}_r'; \vec{x}_s')}$$

$$\times \exp \left\{ -i\omega \left[ T(\vec{x}_r, \vec{y}^S) + T(\vec{x}_s, \vec{y}^S) \right] + i\omega' \left[ T(\vec{x}_r', \vec{y}^S) + T(\vec{x}_s', \vec{y}^S) \right] \right\}$$

It forms the image with the superposition of the backpropagated local cross correlations of the traces.

Coherent Interferometric Imaging (CINT):

\[
\mathcal{I}_{\text{CINT}}(\vec{y}^S) = \sum_{s,s'=1}^{N_s} \sum_{r,r'=1}^{N_r} \int \int \ d\omega d\omega' \hat{u}(\omega, \vec{x}_r; \vec{x}_s) \hat{u}(\omega', \vec{x}_r', \vec{x}_s') \\
\times \exp \left\{ -i\omega \left[ \mathcal{T}(\vec{x}_r, \vec{y}^S) + \mathcal{T}(\vec{x}_s, \vec{y}^S) \right] + i\omega' \left[ \mathcal{T}(\vec{x}_r', \vec{y}^S) + \mathcal{T}(\vec{x}_s', \vec{y}^S) \right] \right\}
\]

- Cross-range resolution: \( \lambda L/X_d \) (for \( X_d < a \)). Range resolution: \( c_0/\Omega_d \) (for \( \Omega_d < B \)).
- Statistical stability

\[
\frac{\text{Var}(\mathcal{I}_{\text{CINT}}(\vec{y}^S))}{\mathbb{E}[\mathcal{I}_{\text{CINT}}(\vec{y}^S)]^2} < 1 \text{ when } \frac{X_d}{X_c} < 1, \frac{a}{X_c} > 1 \text{ and/or } \frac{\Omega_d}{\Omega_c} < 1, \frac{B}{\Omega_c} > 1
\]

where \( \Omega_c \) is the decoherence frequency (frequency gap beyond which the frequency components of the recorded signals are not correlated) and \( X_c \) is the decoherence length (distance between sensors beyond which the signals are not correlated).

- The optimal values for the parameters \( \Omega_d \) and \( X_d \) are \( \Omega_c \) and \( X_c \) (can be found by a statistical analysis that depends on the propagation regime).
- An adaptive procedure for estimating optimally the parameters \( \Omega_d \) and \( X_d \) is based on the minimization of a suitable norm of the image.

Numerical simulations (in strongly scattering medium)

Top: computational setup.
Bottom left: image obtained with Kirchhoff Migration using the surface array.
Bottom right: image obtained with CINT using the surface array.

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Use of an auxiliary passive array

Imaging below an "overburden"
From van der Neut and Bakulin (2009)

Imaging below a strong interface
From Mehta et al (2007)

Figure 11. Illustration showing the geometry of the Mars field OBC data acquisition. There are 120 receivers spaced every 50 m on the seafloor and 364 air guns (spaced every 25 m) are fired near the sea surface. Water depth is 1 km.
Imaging below an overburden: problem

Use of a secondary passive array. $\mathbf{x}_s$ is a source, $\mathbf{x}_q$ is a receiver located below the scattering medium. Data: $\{u(t, \mathbf{x}_q; \mathbf{x}_s), q = 1, \ldots, N_q, s = 1, \ldots, N_s\}$.

If the overburden is scattering, then Kirchhoff Migration does not work:

$$\mathcal{I}_{\text{KM}}(\mathbf{y}^S) = \sum_{q=1}^{N_q} \sum_{s=1}^{N_s} u(\mathcal{T}(\mathbf{x}_s, \mathbf{y}^S) + \mathcal{T}(\mathbf{y}^S, \mathbf{x}_q), \mathbf{x}_q; \mathbf{x}_s)$$
Numerical simulations

Top: computational setup.
Left: image obtained with Kirchhoff Migration using the surface array.
Right: image obtained with Kirchhoff Migration using the bottom array.

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Imaging below an overburden: proposed solution

\[ \vec{x}_s \text{ is a source, } \vec{x}_q \text{ is a receiver. Data: } \{ u(t, \vec{x}_q; \vec{x}_s), q = 1, \ldots, N_q, s = 1, \ldots, N_s \}. \]

Image with Kirchhoff Migration of the cross correlation matrix:

\[ I(\vec{y}^S) = \sum_{q,q'=1}^{N_q} C(\mathcal{T}(\vec{x}_q, \vec{y}^S) + \mathcal{T}(\vec{y}^S, \vec{x}_q'), \vec{x}_q, \vec{x}_{q'}), \]

with

\[ C(\tau, \vec{x}_q, \vec{x}_{q'}) = \sum_{s=1}^{N_s} \int u(t, \vec{x}_q; \vec{x}_s)u(t + \tau, \vec{x}_{q'}; \vec{x}_s)dt, \quad q, q' = 1, \ldots, N_q \]

Functional proposed by Bakulin and Calvert (virtual source method) [1].  
- Analogy: imaging with ambient noise.  
- Main idea: The cross correlation is related to the Green’s function.

Imaging below an overburden: proposed solution

\( \vec{x}_s \) is a source, \( \vec{x}_q \) is a receiver. Data: \( \{u(t, \vec{x}_q; \vec{x}_s), q = 1, \ldots, N_q, s = 1, \ldots, N_s\} \).

Image with Kirchhoff Migration of the cross correlation matrix:

\[
I(\vec{y}^S) = \sum_{q,q'=1}^{N_q} C(\mathcal{T}(\vec{x}_q, \vec{y}^S) + \mathcal{T}(\vec{y}^S, \vec{x}_q'), \vec{x}_q, \vec{x}_q'),
\]

with

\[
C(\tau, \vec{x}_q, \vec{x}_q') = \sum_{s=1}^{N_s} \int u(t, \vec{x}_q; \vec{x}_s)u(t + \tau, \vec{x}_q'; \vec{x}_s)dt, \quad q, q' = 1, \ldots, N_q
\]

It is a "special" CINT functional:

\[
I(\vec{y}^S) = \frac{1}{2\pi} \sum_{s=1}^{N_s} \sum_{q,q'=1}^{N_q} \int d\omega \hat{u}(\omega, \vec{x}_q; \vec{x}_s)\hat{u}(\omega, \vec{x}_q'; \vec{x}_s) \exp \left\{ i\omega [\mathcal{T}(\vec{x}_q, \vec{y}^S) + \mathcal{T}(\vec{y}^S, \vec{x}_q')] \right\}
\]
Proof of concept (in ideal situations)

\[
I(\vec{y}^S) = \sum_{q,q'=1}^{N_q} C(\mathcal{T}(\vec{x}_q, \vec{y}^S) + \mathcal{T}(\vec{y}^S, \vec{x}_{q'}), \vec{x}_q, \vec{x}_{q'}),
\]

with

\[
C(\tau, \vec{x}_q, \vec{x}_{q'}) = \sum_{s=1}^{N_s} \int u(t, \vec{x}_q; \vec{x}_s)u(t + \tau, \vec{x}_{q'}; \vec{x}_s)dt , \quad q, q' = 1, \ldots, N_q
\]

- If the sources are point-like and densely surround the region of interest Ω:

\[
\hat{C}(\omega, \vec{x}_q, \vec{x}_{q'}) \simeq \int_{\partial\Omega} \hat{G}(\omega, \vec{x}_q; \vec{x}_s)\hat{G}(\omega, \vec{x}_{q'}; \vec{x}_s)d\sigma(\vec{x}_s) |\hat{f}(\omega)|^2
\]

where \(\hat{G}(\omega, \vec{x}_q; \vec{x}_s)\) is the time-harmonic Green’s function (with reflector) \((\hat{u}(\omega, \vec{x}_q; \vec{x}_s) = \hat{f}(\omega)\hat{G}(\omega, \vec{x}_q; \vec{x}_s))\).

- By Helmholtz-Kirchhoff identity, we find that, in ideal situations:

\[
\hat{C}(\omega, \vec{x}_q, \vec{x}_{q'}) \simeq \frac{\omega}{c_0} \text{Im}(\hat{G}(\omega, \vec{x}_q; \vec{x}_{q'})) |\hat{f}(\omega)|^2
\]

\(\rightarrow\) the cross correlation of the signals at two receivers \(\vec{x}_q\) and \(\vec{x}_{q'}\) looks like the signal recorded at \(\vec{x}_q\) when \(\vec{x}_{q'}\) is a source.

Therefore, Kirchhoff Migration of the cross correlation matrix should give a good image.

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Analysis (in realistic, scattering situations)

\[ \mathcal{I}(\vec{y}^S) = \sum_{q,q'=1}^{N_q} C(\mathcal{T}(\vec{x}_q, \vec{y}^S) + \mathcal{T}(\vec{y}^S, \vec{x}_{q'}), \vec{x}_q, \vec{x}_{q'}) , \]

with

\[ C(\tau, \vec{x}_q, \vec{x}_{q'}) = \sum_{s=1}^{N_s} \int u(t, \vec{x}_q; \vec{x}_s) u(t + \tau, \vec{x}_{q'}; \vec{x}_s) dt , \quad q, q' = 1, \ldots, N_q \]

• Does the imaging function give good images in realistic situations?
  \( \rightarrow \) It is possible to analyze the resolution and stability of the imaging function in randomly scattering media.

• Analysis of several situations (with sources everywhere at the surface) [1]:
  - weakly scattering, isotropic random medium (paraxial regime),
  - strongly scattering, randomly layered medium,
  - strong deterministic interface.
  \( \rightarrow \) The effect of the random medium is canceled.

Analysis (in realistic, scattering situations)

\[
\mathcal{I}(\vec{y}^S) = \sum_{q,q' = 1}^{N_q} C(\mathcal{I}(\vec{x}_q, \vec{y}^S) + \mathcal{I}(\vec{y}^S, \vec{x}_{q'}), \vec{x}_q, \vec{x}_{q'}),
\]

with

\[
C(\tau, \vec{x}_q, \vec{x}_{q'}) = \sum_{s=1}^{N_s} \int u(t, \vec{x}_q; \vec{x}_s) u(t + \tau, \vec{x}_{q'}; \vec{x}_s) dt, \quad q, q' = 1, \ldots, N_q
\]

• Does the imaging function give good images in realistic situations?

\[\rightarrow\] It is possible to analyze the resolution and stability of the imaging function in randomly scattering media.

• Analysis of several situations (with sources everywhere at the surface) [1]:
  - weakly scattering, isotropic random medium (paraxial regime),
  - strongly scattering, randomly layered medium,
  - strong deterministic interface.

\[\rightarrow\] The effect of the random medium is canceled.

• Question: role of scattering when sources are not everywhere?

\[\rightarrow\] Analysis in the random paraxial regime, in the randomly layered regime, in the radiative transfer regime.

Weakly scattering, isotropic random medium

• Random medium model:
  \[ \frac{1}{c^2(x)} = \frac{1}{c_0^2} \left( 1 + \mu(x) \right) \]

\( c_0 \) is a reference speed,
\( \mu(x) \) is a zero-mean random process.
Imaging below an overburden: analysis in the paraxial regime

• Consider the time-harmonic form of the scalar wave equation ($\vec{x} = (x, z)$)

$$
(\partial_z^2 + \Delta_\perp)\hat{u} + \frac{\omega^2}{c_0^2}(1 + \mu(x, z))\hat{u} = 0.
$$

Consider the paraxial regime $\lambda \ll l_c \ll L$. More precisely, in the scaled regime

$$
\omega \to \frac{\omega}{\varepsilon^4}, \quad \mu(x, z) \to \varepsilon^3 \mu\left(\frac{x}{\varepsilon^2}, \frac{z}{\varepsilon^2}\right),
$$

the function $\hat{\phi}^\varepsilon$ defined by

$$
\hat{u}^\varepsilon(\omega, x, z) = e^{i \frac{\omega z}{c_0^2}} \hat{\phi}^\varepsilon\left(\frac{\omega}{\varepsilon^4}, \frac{x}{\varepsilon^2}, z\right)
$$

satisfies

$$
\varepsilon^4 \partial_z^2 \hat{\phi}^\varepsilon + \left(2i \frac{\omega}{c_0} \partial_z \hat{\phi}^\varepsilon + \Delta_\perp \hat{\phi}^\varepsilon + \frac{\omega^2}{c_0^2} \frac{1}{\varepsilon} \mu(x, \frac{z}{\varepsilon^2}) \hat{\phi}^\varepsilon\right) = 0.
$$

• In the regime $\varepsilon \ll 1$, the forward-scattering approximation in direction $z$ is valid and $\hat{\phi} = \lim_{\varepsilon \to 0} \hat{\phi}^\varepsilon$ satisfies the Itô-Schrödinger equation [1]

$$
2i \frac{\omega}{c_0} \partial_z \hat{\phi} + \Delta_\perp \hat{\phi} + \frac{\omega^2}{c_0^2} \dot{B}(x, z) \hat{\phi} = 0
$$

with $B(x, z)$ Brownian field $\mathbb{E}[B(x, z)B(x', z')] = \gamma(x - x') \min(|z|, |z'|)$,

$$
\gamma(x) = \int_{-\infty}^{\infty} \mathbb{E}[\mu(0, 0)\mu(x, z)]dz.
$$

Imaging below an overburden: analysis in the paraxial regime

• Consider the time-harmonic form of the scalar wave equation \((\vec{x} = (x, z))\)

\[(\partial_z^2 + \Delta_\perp)u + \frac{\omega^2}{c_0^2} (1 + \mu(x, z)) u = 0.\]

Consider the paraxial regime \(\lambda \ll l_c \ll L\). More precisely, in the scaled regime

\[\omega \to \frac{\omega}{\varepsilon^4}, \quad \mu(x, z) \to \varepsilon^3 \mu(\frac{x}{\varepsilon^2}, \frac{z}{\varepsilon^2}),\]

the function \(\hat{\phi}^\varepsilon\) defined by

\[\hat{u}^\varepsilon(\omega, x, z) = e^{i\frac{\omega z}{c_0^4 \varepsilon}} \hat{\phi}^\varepsilon(\frac{\omega}{\varepsilon^4}, \frac{x}{\varepsilon^2}, z)\]

satisfies

\[\varepsilon^4 \partial_z^2 \hat{\phi}^\varepsilon + \left(2i\frac{\omega}{c_0} \partial_z \hat{\phi}^\varepsilon + \Delta_\perp \hat{\phi}^\varepsilon + \frac{\omega^2}{c_0^2} \frac{1}{\varepsilon} \mu(x, \frac{z}{\varepsilon^2}) \hat{\phi}^\varepsilon \right) = 0.\]

• In the regime \(\varepsilon \ll 1\), the forward-scattering approximation in direction \(z\) is valid and \(\hat{\phi} = \lim_{\varepsilon \to 0} \hat{\phi}^\varepsilon\) satisfies the Itô-Schrödinger equation [1]

\[d\hat{\phi} = \frac{ic_0}{2\omega} \Delta_\perp \hat{\phi} dz + \frac{i\omega}{2c_0} \hat{\phi} \circ dB(x, z)\]

with \(B(x, z)\) Brownian field \(\mathbb{E}[B(x, z)B(x', z')] = \gamma(x - x') \min(|z|, |z'|),\)

\(\gamma(x) = \int_{-\infty}^{\infty} \mathbb{E}[\mu(0, 0)\mu(x, z)]dz.\)

We introduce the fundamental solution $\hat{G}(\omega, (\mathbf{x}, z), (\mathbf{x}_0, z_0))$:

$$d\hat{G} = \frac{ic_0}{2\omega} \Delta_{\perp} \hat{G}dz + \frac{i\omega}{2c_0} \hat{G} \circ dB(\mathbf{x}, z)$$

starting from $\hat{G}(\omega, (\mathbf{x}, z = z_0), (\mathbf{x}_0, z_0)) = \delta(\mathbf{x} - \mathbf{x}_0)$.

In a homogeneous medium ($\mu \equiv 0, B \equiv 0$) the fundamental solution is

$$\hat{G}_0(\omega, (\mathbf{x}, z), (\mathbf{x}_0, z_0)) = \exp\left(\frac{i\omega |\mathbf{x} - \mathbf{x}_0|^2}{2c_0|z - z_0|}\right).$$

In a random medium:

$$\mathbb{E}[\hat{G}(\omega, (\mathbf{x}, z), (\mathbf{x}_0, z_0))] = \hat{G}_0(\omega, (\mathbf{x}, z), (\mathbf{x}_0, z_0)) \exp\left(-\frac{\gamma(0)\omega^2|z - z_0|}{8c_0^2}\right),$$

where $\gamma(\mathbf{x}) = \int_{-\infty}^\infty \mathbb{E}[\mu(0, 0)\mu(\mathbf{x}, z)]dz \implies$ Strong damping of the coherent wave $\implies$ Coherent imaging methods fail.

$$\mathbb{E}[\hat{G}(\omega, (\mathbf{x}, z), (\mathbf{x}_0, z_0))\hat{G}(\omega, (\mathbf{x}', z), (\mathbf{x}_0, z_0))]$$

$$= \hat{G}_0(\omega, (\mathbf{x}, z), (\mathbf{x}_0, z_0))\hat{G}_0(\omega, (\mathbf{x}', z), (\mathbf{x}_0, z_0)) \exp\left(-\frac{\gamma_2(\mathbf{x} - \mathbf{x}')\omega^2|z - z_0|}{4c_0^2}\right),$$

where $\gamma_2(\mathbf{x}) = \int_0^1 \gamma(0) - \gamma(\mathbf{x}s)ds \text{ (note } \gamma_2(0) = 0) \implies$ Lateral decoherence.
- Assume that:
  - the source aperture is $b$ and the receiver aperture is $a$ (use continuum approximation for the source and receiver arrays).
  - there is a point reflector at $\bar{y} = (y, -L_y)$ (use the Born approximation for the reflector).
  - the covariance function $\gamma$ can be expanded as $\gamma(\mathbf{x}) = \gamma(0) - \bar{\gamma}_2 |\mathbf{x}|^2 + o(|\mathbf{x}|^2)$ for $|\mathbf{x}| \ll l_c$.
  - scattering is strong: $\frac{\gamma(0) \omega_0^2 L}{c_0^2} > 1$.
- There are two critical lengths:

$$a_{\text{eff}} = b_{\text{eff}} \frac{L_y - L}{L_y}, \quad b_{\text{eff}}^2 = b^2 + \frac{\bar{\gamma}_2 L^3}{3}$$
Homogeneous medium

Random medium

Effective source aperture:

\[ b_{\text{eff}}^2 = b^2 + \frac{\bar{\gamma}_2 L^3}{3} \]
Homogeneous medium Random medium

Effective source and receiver apertures:

\[ a_{\text{eff}} = b_{\text{eff}} \frac{L_y - L}{L_y}, \quad b_{\text{eff}}^2 = b^2 + \frac{\tilde{\gamma}_2 L^3}{3} \]
Migration of the cross correlation matrix

- The Kirchhoff Migration function for the search point $\vec{y}^S$ is

$$I(\vec{y}^S) = \frac{1}{N_d^2} \sum_{q,q'=1}^{N_q} C\left(\frac{\left|\vec{x}_q - \vec{y}^S\right| + \left|\vec{y}^S - \vec{x}_{q'}\right|}{c_0}, \vec{x}_q, \vec{x}_{q'}\right)$$

- The imaging function is statistically stable ($\lambda \ll b \ll L$).

- The cross range resolution is $\frac{\lambda_0 (L_y - L)}{a_{\text{eff}}}$.

The range resolution is $\frac{c_0}{B}$.

- Since $a_{\text{eff}} \mid_{\text{rand}} > a_{\text{eff}} \mid_{\text{homo}}$, this shows that scattering helps (it enhances the angular diversity of the illumination) ! (already noticed for time reversal experiments)
Randomly layered medium

- Random medium model ($\vec{x} = (x, z)$):
  \[
  \frac{1}{c^2(\vec{x})} = \frac{1}{c_0^2} (1 + \mu(z))
  \]

$c_0$ is a reference speed,

$\mu(z)$ is a zero-mean random process.
Imaging below an overburden: analysis in the layered regime

- Consider the time-harmonic form of the scalar wave equation (\( \vec{x} = (x, z) \))

\[
(\partial_z^2 + \Delta_\perp)\hat{u} + \frac{\omega^2}{c_0^2}(1 + \mu(z))\hat{u} = 0
\]

Consider the regime \( l_c \ll \lambda \ll L \), more precisely, the scaled regime

\[
\omega \to \frac{\omega}{\varepsilon}, \quad \mu(z) \to \mu\left(\frac{z}{\varepsilon^2}\right)
\]

- For a point source located at \( \vec{x}_s = (x_s, 0) \) emitting the pulse \( f(t) \), a receiver located at \( \vec{x}_q = (x_q, -L) \), the transmitted field is

\[
u(t, \vec{x}_q; \vec{x}_s) = -\frac{1}{(2\pi)^3} \int_R d\omega \iint_{R^2} \omega^2 d\kappa \hat{f}(\omega) G_{\omega, \kappa} \exp \left( -i\omega(t - \kappa \cdot (x_q - x_s) - \frac{L}{c_0(\kappa)} ) \right)
\]

- we use a Fourier transform in time and transverse spatial coordinates.
- \( c_0(\kappa) \) is the mode-dependent velocity:

\[
c_0(\kappa) = \frac{c_0}{\sqrt{1 - \kappa^2 c_0^2}}
\]

- \( G_{\omega, \kappa} \) is the random Green’s function (transmission coefficient) whose moments are known [1].

Propagation through a randomly layered overburden: analysis (1/2)

\[ G_{\omega, \kappa} = \sum_{j=0}^{\infty} T_{\omega, \kappa}(R_{\omega, \kappa})^j \]

where \( T_{\omega, \kappa} \) and \( R_{\omega, \kappa} \) are the transmission and reflection coefficients for the random slab in \((-L, 0)\) (we have \( |T_{\omega, \kappa}|^2 + |R_{\omega, \kappa}|^2 = 1 \)) [1].

In a homogeneous medium \( G_{\omega, \kappa} \) is equal to 1 because \( T_{\omega, \kappa} = 1 \) and \( R_{\omega, \kappa} = 0 \).

Propagation through a randomly layered overburden: analysis (2/2)

• In a random medium:

\[
\mathbb{E}[T_{\omega,\kappa}] = \exp\left( -\frac{L}{L_{\text{loc}}(\omega, \kappa)} \right)
\]

\[
\mathbb{E}[|T_{\omega,\kappa}|^2] \sim \exp\left( -\frac{L}{4L_{\text{loc}}(\omega, \kappa)} \right)
\]

\[
\mathcal{G}_{\omega,\kappa} = \sum_{j=0}^{\infty} T_{\omega,\kappa}(R_{\omega,\kappa})^j
\]

\[
\mathbb{E}[|\mathcal{G}_{\omega,\kappa}|^2] = 1
\]

• \(\mathbb{E}[T_{\omega,\kappa}] \ll 1\), i.e. most of the energy is in the incoherent fluctuations \(\implies\) Coherent imaging methods fail.

• Exponential decay of \(\mathbb{E}[|T_{\omega,\kappa}|^2]\) specific to randomly layered media \(\implies\) Transmitted signals are very long.

• \(\mathbb{E}[|\mathcal{G}_{\omega,\kappa}|^2] = 1\) \(\implies\) Good (but incoherent) illumination.

• The second-order moment \(\mathbb{E}[\mathcal{G}_{\omega,\kappa}\overline{\mathcal{G}_{\omega,\kappa}'}]\) is given in terms of a a transport-type equation.

• Assume that:
  - the source aperture is $b$ and the receiver aperture is $a$ (use continuum approximation for the source and receiver arrays).
  - there is a point reflector at $\vec{y} = (y, -L_y)$ (use the Born approximation for the reflector).
  - the localization length $L_{loc}$ is smaller than $L$ (strong scattering):
    \[ L_{loc} = \frac{4c_0^2}{\gamma \omega_0^2}, \quad \gamma = \int_{-\infty}^{\infty} \mathbb{E}[\mu(0)\mu(z)]dz \]
• There are two critical lengths:
  \[ a_{eff} = b_{eff} \frac{L_y - L}{L_y}, \quad b_{eff}^2 = 4L_{loc}L \]
Homogeneous medium

Randomly layered medium

Effective source aperture:

\[ b_{\text{eff}} = b \]

\[ b_{\text{eff}}^2 = 4L_{\text{loc}}L \ (\ll b^2) \]
Homogeneous medium

Randomly layered medium

Effective source aperture:

\[ b_{\text{eff}} = b \]

\[ b_{\text{eff}}^2 = 4L_{\text{loc}}L \]

Effective receiver aperture:

\[ a_{\text{eff}} = b \frac{L_y - L}{L_y} \]

\[ a_{\text{eff}} = b_{\text{eff}} \frac{L_y - L}{L_y} \]
Migration of the cross correlation matrix

- The Kirchhoff Migration function for the search point $\vec{y}^S$ is
  \[ I(\vec{y}^S) = \frac{1}{N^2} \sum_{q,q'=1}^{N_q} C\left(\frac{|\vec{x}_q - \vec{y}^S| + |\vec{y}^S - \vec{x}_{q'}|}{c_0}, \vec{x}_q, \vec{x}_{q'}\right) \]

- The imaging function is statistically stable ($\lambda \ll b, L$).

- The cross range resolution is $\frac{\lambda_0 (L_y - L)}{a_{\text{eff}}}$.

  The range resolution is $\frac{c_0}{B} \left(1 + \frac{B^2 L}{4 \omega_0^2 L_{\text{loc}}}\right)^{1/2}$.

- Since $a_{\text{eff}} |_{\text{rand}} < a_{\text{eff}} |_{\text{homo}}$, this shows that scattering does not help (it reduces the angular diversity of the illumination)!
Numerical simulations in a strongly scattering medium
Numerical simulations

Top: computational setup.
Left: image obtained with Kirchhoff Migration using the surface array.
Middle: image obtained with Kirchhoff Migration using the bottom array.
Right: image obtained with the cross correlation technique using the bottom array.

Cargèse April 24, 2013
Conclusions

• Ideal situation for the cross correlation technique (with active sources everywhere):

What is the role of scattering if the sources are spatially localized?
The answer depends on the scattering regime:
- in the isotropic case, random scattering helps (enhances the source aperture).
- in the layered case, random scattering is bad (reduces the source aperture).

• Same conclusion for the $C^3$ technique.

• Here the medium was assumed to be homogeneous in the underburden (between the secondary array and the reflector).
What happens if it is scattering? Modify the cut-off parameters of the CINT functional (for weakly scattering underburden).
Perspectives

- Space surveillance and imaging with airborne passive synthetic aperture arrays.