Role of scattering in correlation-based imaging in random media

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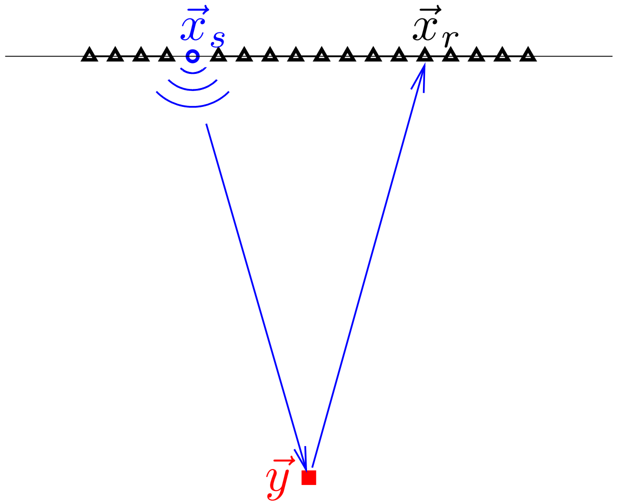
Active imaging through a homogeneous medium

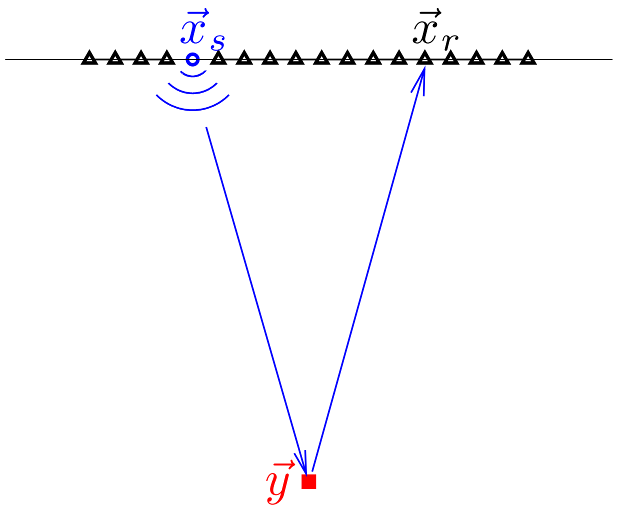


• Sensor array imaging of a reflector located at $\vec{\boldsymbol{y}}$. $\vec{\boldsymbol{x}}_s$ is a source, $\vec{\boldsymbol{x}}_r$ is a receiver.

Data: $\{u(t, \vec{\boldsymbol{x}}_r; \vec{\boldsymbol{x}}_s), r = 1, \ldots, N_\mathrm{r}, s = 1, \ldots, N_\mathrm{s}\}$.

$$\frac{1}{c_0^2}\big(1 + \sigma_\mathrm{ref}\mathbf{1}_{B_\mathrm{ref}}(\vec{\boldsymbol{x}} - \vec{\boldsymbol{y}})\big)\frac{\partial^2 u}{\partial t^2}(t, \vec{\boldsymbol{x}}; \vec{\boldsymbol{x}}_\mathrm{s}) - \Delta_{\vec{\boldsymbol{x}}} u(t, \vec{\boldsymbol{x}}; \vec{\boldsymbol{x}}_\mathrm{s}) = f(t)\delta(\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}_\mathrm{s})$$

Active imaging through a homogeneous medium



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$$\frac{1}{c_0^2}\big(1 + \sigma_{\mathrm{ref}} \mathbf{1}_{B_{\mathrm{ref}}}(\vec{\boldsymbol{x}} - \vec{\boldsymbol{y}})\big)\frac{\partial^2 u}{\partial t^2}(t, \vec{\boldsymbol{x}}; \vec{\boldsymbol{x}}_{\mathrm{s}}) - \Delta_{\vec{\boldsymbol{x}}} u(t, \vec{\boldsymbol{x}}; \vec{\boldsymbol{x}}_{\mathrm{s}}) = f(t)\delta(\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}_{\mathrm{s}})$$

- Image with Kirchhoff Migration:

$$\mathcal{I}_{\mathrm{KM}}(\vec{\boldsymbol{y}}^S) = \sum_{r=1}^{N_{\mathrm{r}}} \sum_{s=1}^{N_{\mathrm{s}}} u\big(\mathcal{T}(\vec{\boldsymbol{x}}_s, \vec{\boldsymbol{y}}^S) + \mathcal{T}(\vec{\boldsymbol{y}}^S, \vec{\boldsymbol{x}}_r), \vec{\boldsymbol{x}}_r; \vec{\boldsymbol{x}}_s\big)$$

It forms the image with the superposition of the backpropagated traces.
$\mathcal{T}(\vec{\boldsymbol{y}}^S, \vec{\boldsymbol{x}})$ is the travel time from $\vec{\boldsymbol{x}}$ to $\vec{\boldsymbol{y}}^S$, i.e. $\mathcal{T}(\vec{\boldsymbol{y}}^S, \vec{\boldsymbol{x}}) = |\vec{\boldsymbol{y}}^S - \vec{\boldsymbol{x}}|/c_0$.

Kirchhoff Migration:

$$\mathcal{I}_{\mathrm{KM}}(\vec{\boldsymbol{y}}^S) = \sum_{r=1}^{N_\mathrm{r}} \sum_{s=1}^{N_\mathrm{s}} u\big(\mathcal{T}(\vec{\boldsymbol{x}}_s, \vec{\boldsymbol{y}}^S) + \mathcal{T}(\vec{\boldsymbol{y}}^S, \vec{\boldsymbol{x}}_r), \vec{\boldsymbol{x}}_r; \vec{\boldsymbol{x}}_s\big)$$

$$\mathcal{I}_{\mathrm{KM}}(\vec{\boldsymbol{y}}^S) = \frac{1}{2\pi} \sum_{r=1}^{N_\mathrm{r}} \sum_{s=1}^{N_\mathrm{s}} \int \overline{\hat{u}(\omega, \vec{\boldsymbol{x}}_r; \vec{\boldsymbol{x}}_s)} \exp\Big\{ i\omega \big[\mathcal{T}(\vec{\boldsymbol{x}}_s, \vec{\boldsymbol{y}}^S) + \mathcal{T}(\vec{\boldsymbol{y}}^S, \vec{\boldsymbol{x}}_r)\big]\Big\} d\omega$$

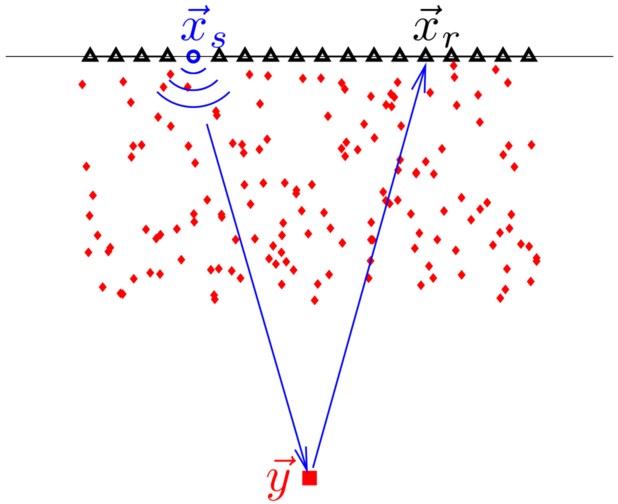
- When $\lambda \ll a \ll L$:

Cross-range resolution: $\lambda L/a$, where λ is the central wavelength, L is the distance from the array to the reflector, and a is the array diameter.
Range resolution: c_0/B, where c_0 is the background velocity and B is the bandwidth.

- Very robust with respect to additive measurement noise.

- Sensitive to clutter noise (scattering medium): If the medium is scattering, then Kirchhoff Migration (usually) does not work.

Imaging through a weakly scattering medium



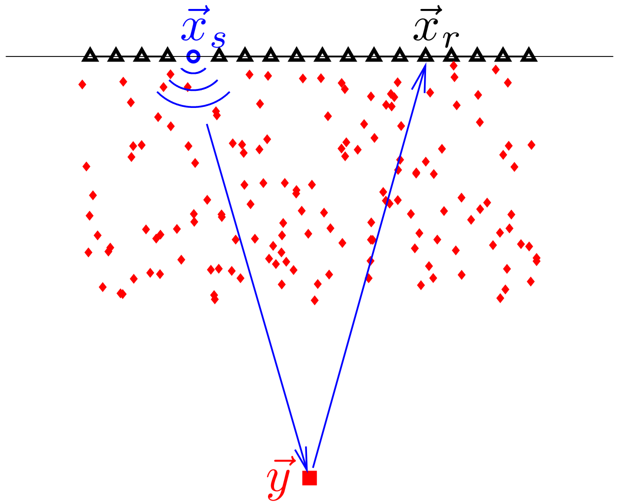
Sensor array imaging of a reflector located at $\vec{\boldsymbol{y}}$. $\vec{\boldsymbol{x}}_s$ is a source, $\vec{\boldsymbol{x}}_r$ is a receiver.
Data: $\{u(t,\vec{\boldsymbol{x}}_r;\vec{\boldsymbol{x}}_s), r=1,\dots,N_{\mathrm{r}}, s=1,\dots,N_{\mathrm{s}}\}$.

If the medium is weakly scattering, then Kirchhoff migration does not work:

$$\begin{aligned}
\big|\mathcal{I}_{\mathrm{KM}}(\vec{\boldsymbol{y}}^S)\big|^2 &= \Big|\sum_{s=1}^{N_{\mathrm{s}}}\sum_{r=1}^{N_{\mathrm{r}}}\int \overline{\hat{u}(\omega,\vec{\boldsymbol{x}}_r;\vec{\boldsymbol{x}}_s)}\exp\Big\{i\omega\big[\mathcal{T}(\vec{\boldsymbol{x}}_s,\vec{\boldsymbol{y}}^S)+\mathcal{T}(\vec{\boldsymbol{y}}^S,\vec{\boldsymbol{x}}_r)\big]\Big\}d\omega\Big|^2 \\
&= \sum_{s,s'=1}^{N_{\mathrm{s}}}\sum_{r,r'=1}^{N_{\mathrm{r}}}\iint d\omega d\omega'\,\hat{u}(\omega,\vec{\boldsymbol{x}}_r;\vec{\boldsymbol{x}}_s)\overline{\hat{u}(\omega',\vec{\boldsymbol{x}}_{r'},\vec{\boldsymbol{x}}_{s'})} \\
&\quad\times\exp\Big\{-i\omega\big[\mathcal{T}(\vec{\boldsymbol{x}}_r,\vec{\boldsymbol{y}}^S)+\mathcal{T}(\vec{\boldsymbol{x}}_s,\vec{\boldsymbol{y}}^S)\big]+i\omega'\big[\mathcal{T}(\vec{\boldsymbol{x}}_{r'},\vec{\boldsymbol{y}}^S)+\mathcal{T}(\vec{\boldsymbol{x}}_{s'},\vec{\boldsymbol{y}}^S)\big]\Big\}
\end{aligned}$$

Problem because $\hat{u}(\omega,\vec{\boldsymbol{x}}_r;\vec{\boldsymbol{x}}_s)$ and $\hat{u}(\omega',\vec{\boldsymbol{x}}_{r'},\vec{\boldsymbol{x}}_{s'})$ can be uncorrelated.

Imaging through a weakly scattering medium



Sensor array imaging of a reflector located at $\vec{\boldsymbol{y}}$. $\vec{\boldsymbol{x}}_s$ is a source, $\vec{\boldsymbol{x}}_r$ is a receiver.
Data: $\{u(t, \vec{\boldsymbol{x}}_r; \vec{\boldsymbol{x}}_s), r = 1, \ldots, N_\mathrm{r}, s = 1, \ldots, N_\mathrm{s}\}$.

If the medium is weakly scattering, then image with Coherent Interferometric Imaging (CINT) [1]:

$$\mathcal{I}_{\mathrm{CINT}}(\vec{\boldsymbol{y}}^S) = \sum_{\substack{s,s'=1 \\ |\vec{\boldsymbol{x}}_s - \vec{\boldsymbol{x}}_{s'}| \leq X_\mathrm{d}}}^{N_\mathrm{s}} \sum_{\substack{r,r'=1 \\ |\vec{\boldsymbol{x}}_r - \vec{\boldsymbol{x}}_{r'}| \leq X_\mathrm{d}}}^{N_\mathrm{r}} \iint_{|\omega - \omega'| \leq \Omega_\mathrm{d}} d\omega d\omega' \, \hat{u}(\omega, \vec{\boldsymbol{x}}_r; \vec{\boldsymbol{x}}_s) \overline{\hat{u}(\omega', \vec{\boldsymbol{x}}_{r'}, \vec{\boldsymbol{x}}_{s'})}$$

$$\times \exp\Big\{ -i\omega\big[\mathcal{T}(\vec{\boldsymbol{x}}_r, \vec{\boldsymbol{y}}^S) + \mathcal{T}(\vec{\boldsymbol{x}}_s, \vec{\boldsymbol{y}}^S)\big] + i\omega'\big[\mathcal{T}(\vec{\boldsymbol{x}}_{r'}, \vec{\boldsymbol{y}}^S) + \mathcal{T}(\vec{\boldsymbol{x}}_{s'}, \vec{\boldsymbol{y}}^S)\big]\Big\}$$

It forms the image with the superposition of the backpropagated local cross correlations of the traces.

[1] L. Borcea, G. Papanicolaou, and C. Tsogka, *Inverse Problems* **22**, 1405 (2006).

Coherent Interferometric Imaging (CINT):

$$\mathcal{I}_{\rm CINT}(\vec{\boldsymbol{y}}^S) = \sum_{\substack{s,s'=1 \\ |\vec{\boldsymbol{x}}_s-\vec{\boldsymbol{x}}_{s'}|\leq X_{\rm d}}}^{N_{\rm s}} \sum_{\substack{r,r'=1 \\ |\vec{\boldsymbol{x}}_r-\vec{\boldsymbol{x}}_{r'}|\leq X_{\rm d}}}^{N_{\rm r}} \iint_{|\omega-\omega'|\leq\Omega_{\rm d}} d\omega d\omega' \, \hat{u}(\omega,\vec{\boldsymbol{x}}_r;\vec{\boldsymbol{x}}_s)\overline{\hat{u}(\omega',\vec{\boldsymbol{x}}_{r'},\vec{\boldsymbol{x}}_{s'})}$$

$$\times \exp\Big\{ - i\omega\big[\mathcal{T}(\vec{\boldsymbol{x}}_r,\vec{\boldsymbol{y}}^S) + \mathcal{T}(\vec{\boldsymbol{x}}_s,\vec{\boldsymbol{y}}^S)\big] + i\omega'\big[\mathcal{T}(\vec{\boldsymbol{x}}_{r'},\vec{\boldsymbol{y}}^S) + \mathcal{T}(\vec{\boldsymbol{x}}_{s'},\vec{\boldsymbol{y}}^S)\big]\Big\}$$

- Cross-range resolution: $\lambda L/X_{\rm d}$ (for $X_{\rm d} < a$). Range resolution: $c_0/\Omega_{\rm d}$ (for $\Omega_{\rm d} < B$).
- Statistical stability

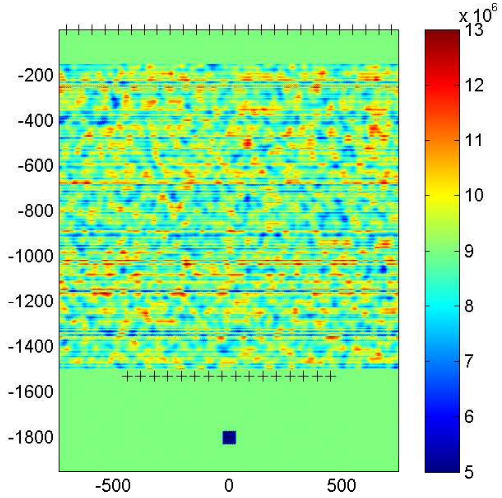
$$\frac{{\rm Var}\big(\mathcal{I}_{\rm CINT}(\vec{\boldsymbol{y}}^S)\big)}{\mathbb{E}\big[\mathcal{I}_{\rm CINT}(\vec{\boldsymbol{y}}^S)\big]^2} < 1 \text{ when } \frac{X_{\rm d}}{X_{\rm c}} < 1, \ \frac{a}{X_{\rm c}} > 1 \text{ and/or } \frac{\Omega_{\rm d}}{\Omega_c} < 1, \ \frac{B}{\Omega_{\rm c}} > 1$$

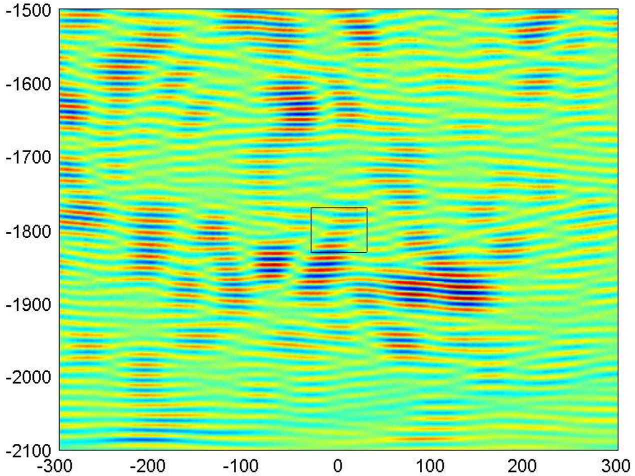
where $\Omega_{\rm c}$ is the decoherence frequency (frequency gap beyond which the frequency components of the recorded signals are not correlated) and $X_{\rm c}$ is the decoherence length (distance between sensors beyond which the signals are not correlated).

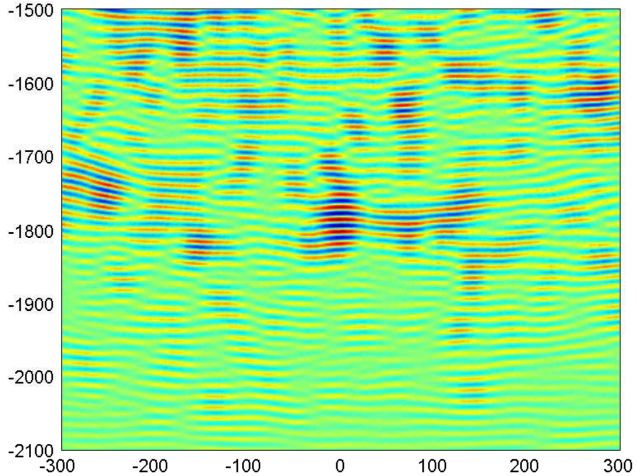
- The optimal values for the parameters $\Omega_{\rm d}$ and $X_{\rm d}$ are $\Omega_{\rm c}$ and $X_{\rm c}$ (can be found by a statistical analysis that depends on the propagation regime).
- An adaptive procedure for estimating optimally the parameters $\Omega_{\rm d}$ and $X_{\rm d}$ is based on the minimization of a suitable norm of the image.

[1] L. Borcea, J. Garnier, G. Papanicolaou, and C. Tsogka, *Inverse Problems* **27**, 085004 (2011).

Numerical simulations (in strongly scattering medium)





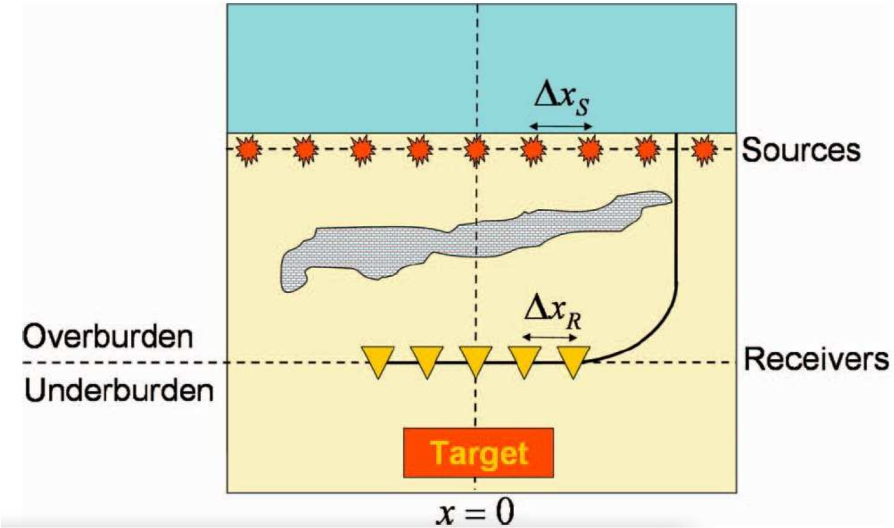


Top: computational setup.

Bottom left: image obtained with Kirchhoff Migration using the surface array.

Bottom right: image obtained with CINT using the surface array.

Use of an auxiliary passive array



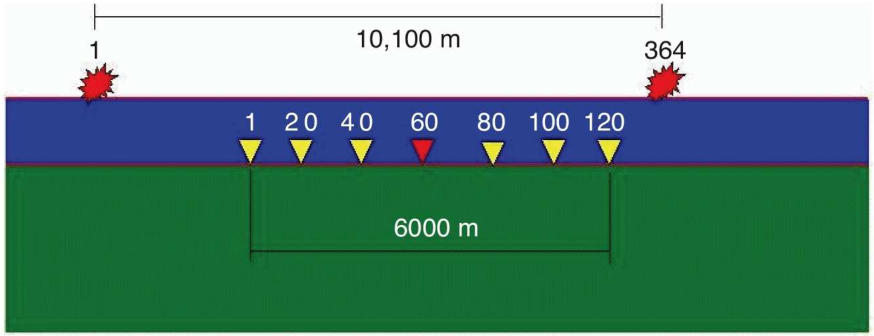


Figure 11. Illustration showing the geometry of the Mars field OBC data acquisition. There are 120 receivers spaced every 50 m on the seafloor and 364 air guns (spaced every 25 m) are fired near the sea surface. Water depth is 1 km.

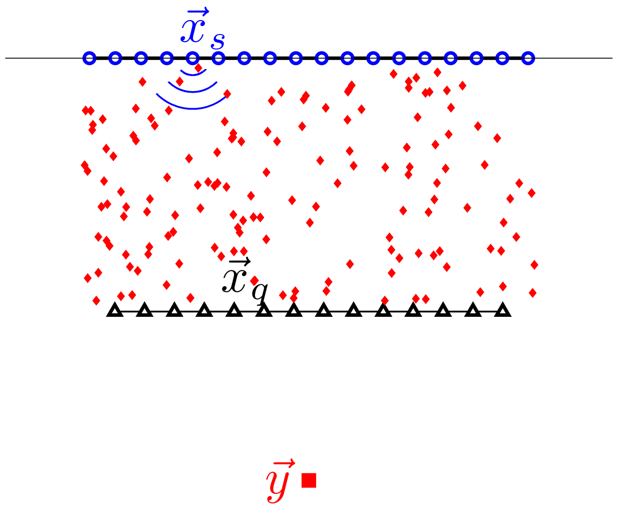
Imaging below an "overburden"

From van der Neut and Bakulin (2009)

Imaging below a strong interface

From Mehta et al (2007)

Imaging below an overburden: problem

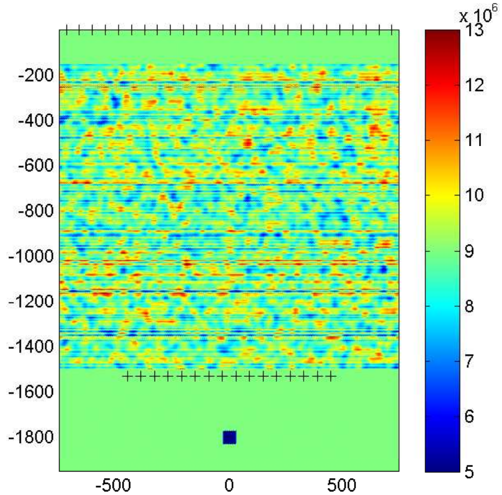


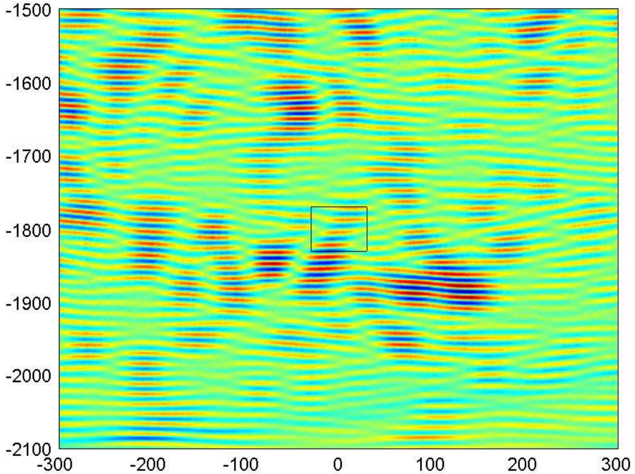
Use of a secondary *passive* array. $\vec{\boldsymbol{x}}_s$ is a source, $\vec{\boldsymbol{x}}_q$ is a receiver located below the scattering medium. Data: $\{u(t, \vec{\boldsymbol{x}}_q; \vec{\boldsymbol{x}}_s), q = 1, \ldots, N_\mathrm{q}, s = 1, \ldots, N_\mathrm{s}\}$.

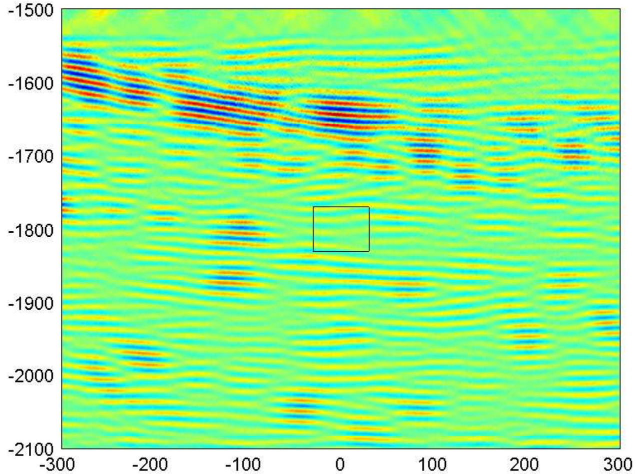
If the overburden is scattering, then Kirchhoff Migration does not work:

$$\mathcal{I}_{\mathrm{KM}}(\vec{\boldsymbol{y}}^S) = \sum_{q=1}^{N_\mathrm{q}} \sum_{s=1}^{N_\mathrm{s}} u\big(\mathcal{T}(\vec{\boldsymbol{x}}_s, \vec{\boldsymbol{y}}^S) + \mathcal{T}(\vec{\boldsymbol{y}}^S, \vec{\boldsymbol{x}}_q), \vec{\boldsymbol{x}}_q; \vec{\boldsymbol{x}}_s\big)$$

Numerical simulations





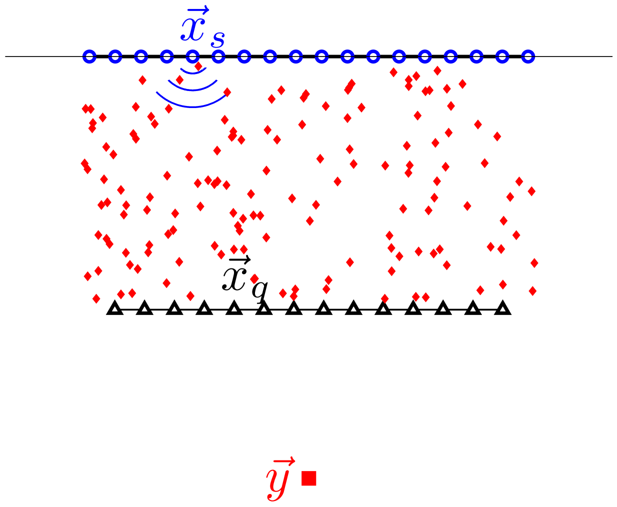


Top: computational setup.

Left: image obtained with Kirchhoff Migration using the surface array.

Right: image obtained with Kirchhoff Migration using the bottom array.

Imaging below an overburden: proposed solution



$\vec{\boldsymbol{x}}_s$ is a source, $\vec{\boldsymbol{x}}_q$ is a receiver. Data: $\{u(t, \vec{\boldsymbol{x}}_q; \vec{\boldsymbol{x}}_s), q = 1, \ldots, N_{\mathrm{q}}, s = 1, \ldots, N_{\mathrm{s}}\}$.

Image with Kirchhoff Migration of the cross correlation matrix:

$$\mathcal{I}(\vec{\boldsymbol{y}}^S) = \sum_{q,q'=1}^{N_{\mathrm{q}}} \mathcal{C}\big(\mathcal{T}(\vec{\boldsymbol{x}}_q, \vec{\boldsymbol{y}}^S) + \mathcal{T}(\vec{\boldsymbol{y}}^S, \vec{\boldsymbol{x}}_{q'}), \vec{\boldsymbol{x}}_q, \vec{\boldsymbol{x}}_{q'}\big),$$

with

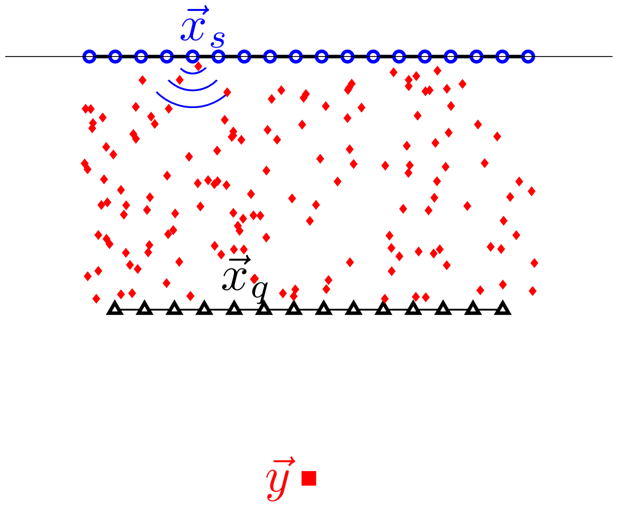
$$\mathcal{C}(\tau, \vec{\boldsymbol{x}}_q, \vec{\boldsymbol{x}}_{q'}) = \sum_{s=1}^{N_{\mathrm{s}}} \int u(t, \vec{\boldsymbol{x}}_q; \vec{\boldsymbol{x}}_s) u(t+\tau, \vec{\boldsymbol{x}}_{q'}; \vec{\boldsymbol{x}}_s) dt \ , \qquad q, q' = 1, \ldots, N_q$$

Functional proposed by Bakulin and Calvert (virtual source method) [1].

- Analogy: imaging with ambient noise.
- Main idea: The cross correlation is related to the Green's function.

[1] A. Bakulin and R. Calvert, *Geophysics* **71** (2006), SI139-SI150.

Imaging below an overburden: proposed solution



$\vec{\boldsymbol{x}}_s$ is a source, $\vec{\boldsymbol{x}}_q$ is a receiver. Data: $\{u(t,\vec{\boldsymbol{x}}_q;\vec{\boldsymbol{x}}_s), q=1,\dots,N_{\rm q}, s=1,\dots,N_{\rm s}\}$.

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with

$$\mathcal{C}(\tau,\vec{\boldsymbol{x}}_q,\vec{\boldsymbol{x}}_{q'}) = \sum_{s=1}^{N_{\rm s}} \int u(t,\vec{\boldsymbol{x}}_q;\vec{\boldsymbol{x}}_s)u(t+\tau,\vec{\boldsymbol{x}}_{q'};\vec{\boldsymbol{x}}_s)dt \ , \qquad q,q'=1,\dots,N_q$$

It is a “special” CINT functional:

$$\mathcal{I}(\vec{\boldsymbol{y}}^S) = \frac{1}{2\pi}\sum_{s=1}^{N_{\rm s}}\sum_{q,q'=1}^{N_{\rm q}} \int d\omega \hat{u}(\omega,\vec{\boldsymbol{x}}_q;\vec{\boldsymbol{x}}_s)\overline{\hat{u}(\omega,\vec{\boldsymbol{x}}_{q'};\vec{\boldsymbol{x}}_s)} \exp\Big\{i\omega\big[\mathcal{T}(\vec{\boldsymbol{x}}_q,\vec{\boldsymbol{y}}^S)+\mathcal{T}(\vec{\boldsymbol{y}}^S,\vec{\boldsymbol{x}}_{q'})\big]\Big\}$$

Proof of concept (in ideal situations)

$$\mathcal{I}(\vec{\boldsymbol{y}}^S) = \sum_{q,q'=1}^{N_\mathrm{q}} \mathcal{C}\big(\mathcal{T}(\vec{\boldsymbol{x}}_q, \vec{\boldsymbol{y}}^S) + \mathcal{T}(\vec{\boldsymbol{y}}^S, \vec{\boldsymbol{x}}_{q'}), \vec{\boldsymbol{x}}_q, \vec{\boldsymbol{x}}_{q'}\big),$$

with

$$\mathcal{C}(\tau, \vec{\boldsymbol{x}}_q, \vec{\boldsymbol{x}}_{q'}) = \sum_{s=1}^{N_\mathrm{s}} \int u(t, \vec{\boldsymbol{x}}_q; \vec{\boldsymbol{x}}_s) u(t+\tau, \vec{\boldsymbol{x}}_{q'}; \vec{\boldsymbol{x}}_s) dt \ , \qquad q, q' = 1, \ldots, N_q$$

- If the sources are point-like and densely surround the region of interest Ω:

$$\hat{\mathcal{C}}(\omega, \vec{\boldsymbol{x}}_q, \vec{\boldsymbol{x}}_{q'}) \simeq \int_{\partial\Omega} \hat{G}(\omega, \vec{\boldsymbol{x}}_q; \vec{\boldsymbol{x}}_s) \overline{\hat{G}(\omega, \vec{\boldsymbol{x}}_{q'}; \vec{\boldsymbol{x}}_s)} d\sigma(\vec{\boldsymbol{x}}_s) \, |\hat{f}(\omega)|^2$$

where $\hat{G}(\omega, \vec{\boldsymbol{x}}_q; \vec{\boldsymbol{x}}_s)$ is the time-harmonic Green's function (with reflector) $(\hat{u}(\omega, \vec{\boldsymbol{x}}_q; \vec{\boldsymbol{x}}_s) = \hat{f}(\omega)\hat{G}(\omega, \vec{\boldsymbol{x}}_q; \vec{\boldsymbol{x}}_s))$.

- By Helmholtz-Kirchhoff identity, we find that, in ideal situations:

$$\hat{\mathcal{C}}(\omega, \vec{\boldsymbol{x}}_q, \vec{\boldsymbol{x}}_{q'}) \simeq \frac{\omega}{c_0} \mathrm{Im}\big(\hat{G}(\omega, \vec{\boldsymbol{x}}_q; \vec{\boldsymbol{x}}_{q'})\big) \, |\hat{f}(\omega)|^2$$

$\hookrightarrow$ the cross correlation of the signals at two receivers $\vec{\boldsymbol{x}}_q$ and $\vec{\boldsymbol{x}}_{q'}$ looks like the signal recorded at $\vec{\boldsymbol{x}}_q$ when $\vec{\boldsymbol{x}}_{q'}$ is a source.
Therefore, Kirchhoff Migration of the cross correlation matrix should give a good image.

Analysis (in realistic, scattering situations)

$$\mathcal{I}(\vec{\boldsymbol{y}}^S) = \sum_{q,q'=1}^{N_\mathrm{q}} \mathcal{C}\big(\mathcal{T}(\vec{\boldsymbol{x}}_q, \vec{\boldsymbol{y}}^S) + \mathcal{T}(\vec{\boldsymbol{y}}^S, \vec{\boldsymbol{x}}_{q'}), \vec{\boldsymbol{x}}_q, \vec{\boldsymbol{x}}_{q'}\big),$$

with

$$\mathcal{C}(\tau, \vec{\boldsymbol{x}}_q, \vec{\boldsymbol{x}}_{q'}) = \sum_{s=1}^{N_\mathrm{s}} \int u(t, \vec{\boldsymbol{x}}_q; \vec{\boldsymbol{x}}_s) u(t+\tau, \vec{\boldsymbol{x}}_{q'}; \vec{\boldsymbol{x}}_s) dt \;, \qquad q, q' = 1, \ldots, N_q$$

• Does the imaging function give good images in realistic situations ?

$\hookrightarrow$ It is possible to analyze the resolution and stability of the imaging function in randomly scattering media.

• Analysis of several situations (with sources everywhere at the surface) [1]:

- weakly scattering, isotropic random medium (paraxial regime),
- strongly scattering, randomly layered medium,
- strong deterministic interface.

$\hookrightarrow$ The effect of the random medium is canceled.

[1] J. Garnier and G. Papanicolaou, *Inverse Problems* **28** 075002 (2012).

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- weakly scattering, isotropic random medium (paraxial regime),
- strongly scattering, randomly layered medium,
- strong deterministic interface.

$\hookrightarrow$ The effect of the random medium is canceled.

• Question: role of scattering when sources are not everywhere ?
$\hookrightarrow$ Analysis in the random paraxial regime, in the randomly layered regime, in the radiative transfer regime.

[1] J. Garnier and G. Papanicolaou, *Inverse Problems* **28** 075002 (2012).

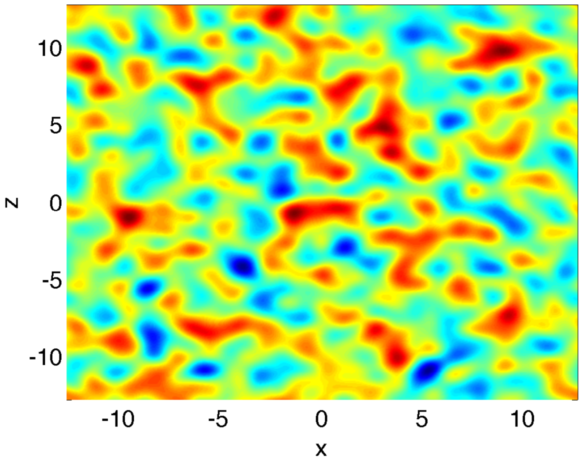
Weakly scattering, isotropic random medium

- Random medium model:

$$\frac{1}{c^2(\vec{\boldsymbol{x}})} = \frac{1}{c_0^2}\big(1 + \mu(\vec{\boldsymbol{x}})\big)$$

c_0 is a reference speed,

$\mu(\vec{\boldsymbol{x}})$ is a zero-mean random process.



Imaging below an overburden: analysis in the paraxial regime

- Consider the time-harmonic form of the scalar wave equation ($\vec{\boldsymbol{x}} = (\boldsymbol{x}, z)$)

$$(\partial_z^2 + \Delta_\perp)\hat{u} + \frac{\omega^2}{c_0^2}\big(1 + \mu(\boldsymbol{x}, z)\big)\hat{u} = 0.$$

Consider the paraxial regime $\lambda \ll l_c \ll L$. More precisely, in the scaled regime

$$\omega \to \frac{\omega}{\varepsilon^4}, \qquad \mu(\boldsymbol{x}, z) \to \varepsilon^3 \mu\big(\frac{\boldsymbol{x}}{\varepsilon^2}, \frac{z}{\varepsilon^2}\big),$$

the function $\hat{\phi}^\varepsilon$ defined by

$$\hat{u}^\varepsilon(\omega, \boldsymbol{x}, z) = e^{i\frac{\omega z}{\varepsilon^4 c_0}} \hat{\phi}^\varepsilon\big(\frac{\omega}{\varepsilon^4}, \frac{\boldsymbol{x}}{\varepsilon^2}, z\big)$$

satisfies

$$\varepsilon^4 \partial_z^2 \hat{\phi}^\varepsilon + \left(2i\frac{\omega}{c_0}\partial_z \hat{\phi}^\varepsilon + \Delta_\perp \hat{\phi}^\varepsilon + \frac{\omega^2}{c_0^2}\frac{1}{\varepsilon}\mu\big(\boldsymbol{x}, \frac{z}{\varepsilon^2}\big)\hat{\phi}^\varepsilon\right) = 0.$$

- In the regime $\varepsilon \ll 1$, the forward-scattering approximation in direction z is valid and $\hat{\phi} = \lim_{\varepsilon\to 0} \hat{\phi}^\varepsilon$ satisfies the Itô-Schrödinger equation [1]

$$2i\frac{\omega}{c_0}\partial_z \hat{\phi} + \Delta_\perp \hat{\phi} + \frac{\omega^2}{c_0^2}\dot{B}(\boldsymbol{x}, z)\hat{\phi} = 0$$

with $B(\boldsymbol{x}, z)$ Brownian field $\mathbb{E}[B(\boldsymbol{x}, z)B(\boldsymbol{x}', z')] = \gamma(\boldsymbol{x} - \boldsymbol{x}') \min(|z|, |z'|)$,
$\gamma(\boldsymbol{x}) = \int_{-\infty}^{\infty} \mathbb{E}[\mu(\boldsymbol{0}, 0)\mu(\boldsymbol{x}, z)]dz$.

[1] J. Garnier and K. Sølna, *Ann. Appl. Probab.* **19**, 318 (2009).

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$$\omega \to \frac{\omega}{\varepsilon^4}, \qquad \mu(\boldsymbol{x}, z) \to \varepsilon^3 \mu\Big(\frac{\boldsymbol{x}}{\varepsilon^2}, \frac{z}{\varepsilon^2}\Big),$$

the function $\hat{\phi}^\varepsilon$ defined by

$$\hat{u}^\varepsilon(\omega, \boldsymbol{x}, z) = e^{i\frac{\omega z}{\varepsilon^4 c_0}} \hat{\phi}^\varepsilon\Big(\frac{\omega}{\varepsilon^4}, \frac{\boldsymbol{x}}{\varepsilon^2}, z\Big)$$

satisfies

$$\varepsilon^4 \partial_z^2 \hat{\phi}^\varepsilon + \left(2i\frac{\omega}{c_0}\partial_z \hat{\phi}^\varepsilon + \Delta_\perp \hat{\phi}^\varepsilon + \frac{\omega^2}{c_0^2}\frac{1}{\varepsilon}\mu\Big(\boldsymbol{x}, \frac{z}{\varepsilon^2}\Big)\hat{\phi}^\varepsilon\right) = 0.$$

- In the regime $\varepsilon \ll 1$, the forward-scattering approximation in direction z is valid and $\hat{\phi} = \lim_{\varepsilon \to 0} \hat{\phi}^\varepsilon$ satisfies the Itô-Schrödinger equation [1]

$$d\hat{\phi} = \frac{ic_0}{2\omega}\Delta_\perp \hat{\phi}\, dz + \frac{i\omega}{2c_0}\hat{\phi} \circ dB(\boldsymbol{x}, z)$$

with $B(\boldsymbol{x}, z)$ Brownian field $\mathbb{E}[B(\boldsymbol{x}, z)B(\boldsymbol{x}', z')] = \gamma(\boldsymbol{x} - \boldsymbol{x}') \min(|z|, |z'|)$,
$\gamma(\boldsymbol{x}) = \int_{-\infty}^{\infty} \mathbb{E}[\mu(\boldsymbol{0}, 0)\mu(\boldsymbol{x}, z)]dz$.

[1] J. Garnier and K. Sølna, *Ann. Appl. Probab.* **19**, 318 (2009).

- We introduce the fundamental solution $\hat{G}\big(\omega,(\boldsymbol{x},z),(\boldsymbol{x}_0,z_0)\big)$:

$$d\hat{G} = \frac{ic_0}{2\omega}\Delta_\perp \hat{G}dz + \frac{i\omega}{2c_0}\hat{G}\circ dB(\boldsymbol{x},z)$$

starting from $\hat{G}\big(\omega,(\boldsymbol{x},z=z_0),(\boldsymbol{x}_0,z_0)\big) = \delta(\boldsymbol{x}-\boldsymbol{x}_0)$.

- In a homogeneous medium ($\mu\equiv 0$, $B\equiv 0$) the fundamental solution is

$$\hat{G}_0\big(\omega,(\boldsymbol{x},z),(\boldsymbol{x}_0,z_0)\big) = \frac{\exp\Big(\frac{i\omega|\boldsymbol{x}-\boldsymbol{x}_0|^2}{2c_0|z-z_0|}\Big)}{2i\pi c_0\frac{|z-z_0|}{\omega}}.$$

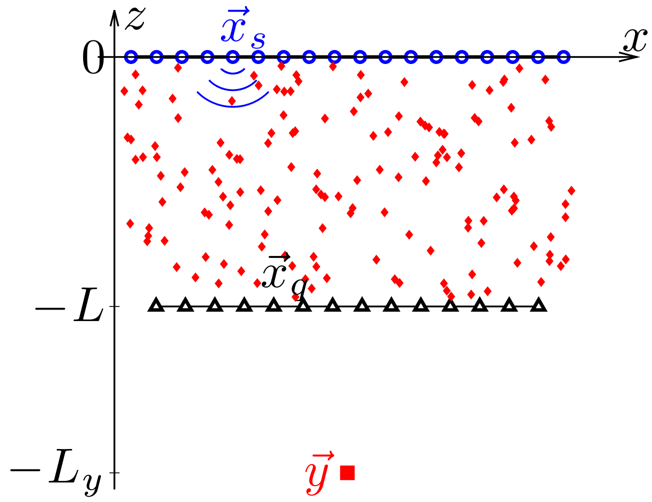
- In a random medium:

$$\mathbb{E}\big[\hat{G}\big(\omega,(\boldsymbol{x},z),(\boldsymbol{x}_0,z_0)\big)\big] = \hat{G}_0\big(\omega,(\boldsymbol{x},z),(\boldsymbol{x}_0,z_0)\big)\exp\Big(-\frac{\gamma(\boldsymbol{0})\omega^2|z-z_0|}{8c_0^2}\Big),$$

where $\gamma(\boldsymbol{x}) = \int_{-\infty}^{\infty}\mathbb{E}[\mu(\boldsymbol{0},0)\mu(\boldsymbol{x},z)]dz \Longrightarrow$ Strong damping of the coherent wave $\Longrightarrow$ Coherent imaging methods fail.

$$\mathbb{E}\big[\hat{G}\big(\omega,(\boldsymbol{x},z),(\boldsymbol{x}_0,z_0)\big)\overline{\hat{G}\big(\omega,(\boldsymbol{x}',z),(\boldsymbol{x}_0,z_0)\big)}\big]$$
$$= \hat{G}_0\big(\omega,(\boldsymbol{x},z),(\boldsymbol{x}_0,z_0)\big)\overline{\hat{G}_0\big(\omega,(\boldsymbol{x}',z),(\boldsymbol{x}_0,z_0)\big)}\exp\Big(-\frac{\gamma_2(\boldsymbol{x}-\boldsymbol{x}')\omega^2|z-z_0|}{4c_0^2}\Big),$$

where $\gamma_2(\boldsymbol{x}) = \int_0^1\gamma(\boldsymbol{0})-\gamma(\boldsymbol{x}s)ds$ (note $\gamma_2(\boldsymbol{0})=0$) $\Longrightarrow$ Lateral decoherence.

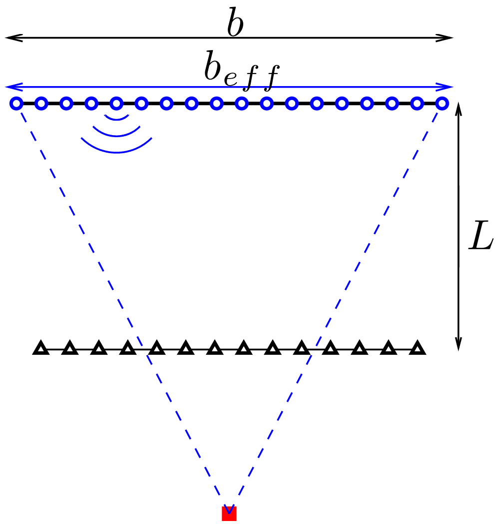


- Assume that:

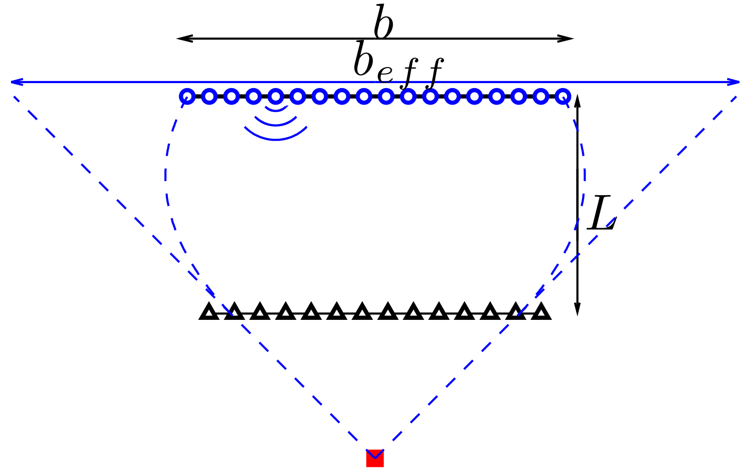
- the source aperture is b and the receiver aperture is a (use continuum approximation for the source and receiver arrays).
- there is a point reflector at $\vec{\boldsymbol{y}} = (\boldsymbol{y}, -L_y)$ (use the Born approximation for the reflector).
- the covariance function γ can be expanded as $\gamma(\boldsymbol{x}) = \gamma(\mathbf{0}) - \bar{\gamma}_2 |\boldsymbol{x}|^2 + o(|\boldsymbol{x}|^2)$ for $|\boldsymbol{x}| \ll l_c$.
- scattering is strong: $\frac{\gamma(\mathbf{0})\omega_0^2 L}{c_0^2} > 1$.

- There are two critical lengths:

$$a_{\text{eff}} = b_{\text{eff}} \frac{L_y - L}{L_y}, \qquad b_{\text{eff}}^2 = b^2 + \frac{\bar{\gamma}_2 L^3}{3}$$



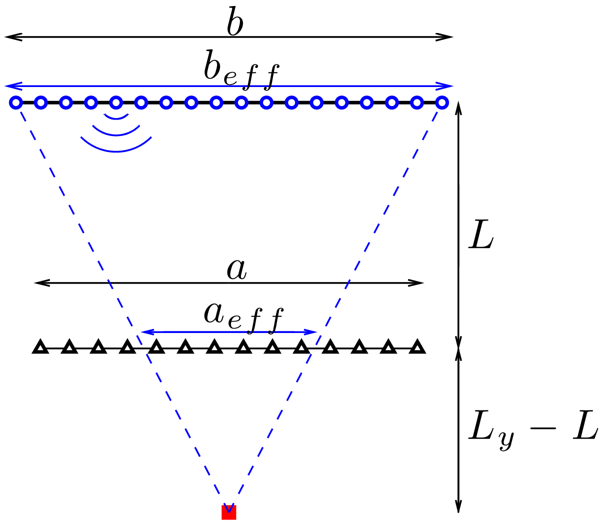
Homogeneous medium



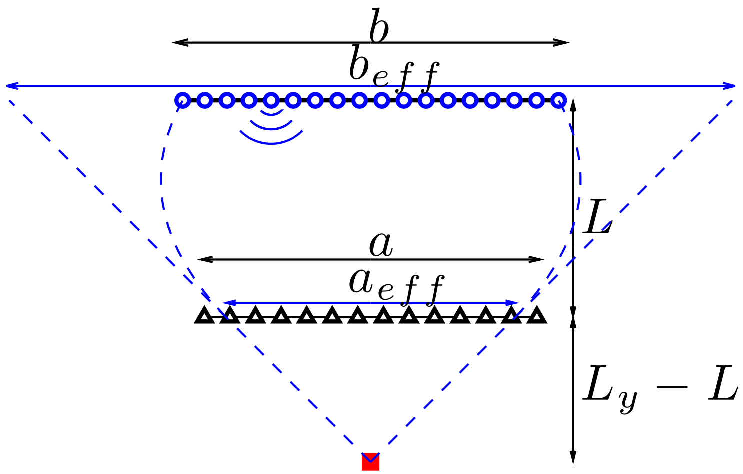
Random medium

Effective source aperture:

$$b_{\text{eff}}^2 = b^2 + \frac{\bar{\gamma}_2 L^3}{3}$$



Homogeneous medium



Random medium

Effective source and receiver apertures:

$$a_{\rm eff} = b_{\rm eff}\frac{L_y - L}{L_y}, \qquad b_{\rm eff}^2 = b^2 + \frac{\bar{\gamma}_2 L^3}{3}$$

Migration of the cross correlation matrix

- The Kirchhoff Migration function for the search point $\vec{\boldsymbol{y}}^S$ is

$$\mathcal{I}(\vec{\boldsymbol{y}}^S) = \frac{1}{N_{\rm q}^2} \sum_{q,q'=1}^{N_{\rm q}} \mathcal{C}\Big(\frac{|\vec{\boldsymbol{x}}_q - \vec{\boldsymbol{y}}^S| + |\vec{\boldsymbol{y}}^S - \vec{\boldsymbol{x}}_{q'}|}{c_0}, \vec{\boldsymbol{x}}_q, \vec{\boldsymbol{x}}_{q'} \Big)$$

- The imaging function is statistically stable ($\lambda \ll b \ll L$).

- The cross range resolution is $\dfrac{\lambda_0 (L_y - L)}{a_{\rm eff}}$.

The range resolution is $\dfrac{c_0}{B}$.

- Since $a_{\rm eff} \mid_{\rm rand} > a_{\rm eff} \mid_{\rm homo}$, this shows that scattering helps (it enhances the angular diversity of the illumination) ! (already noticed for time reversal experiments)

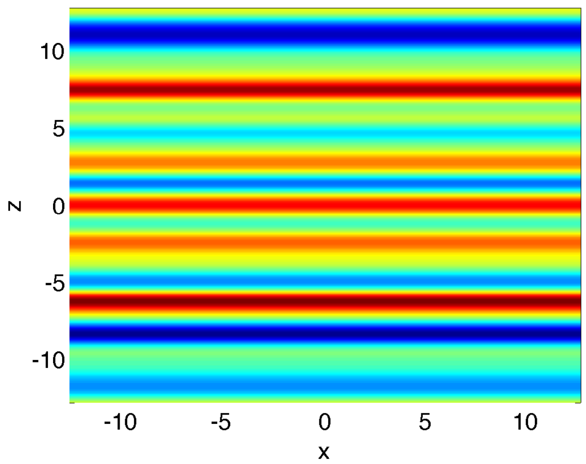
Randomly layered medium

- Random medium model ($\vec{\boldsymbol{x}} = (\boldsymbol{x}, z)$):

$$\frac{1}{c^2(\vec{\boldsymbol{x}})} = \frac{1}{c_0^2}\big(1 + \mu(z)\big)$$

c_0 is a reference speed,

$\mu(z)$ is a zero-mean random process.



Imaging below an overburden: analysis in the layered regime

- Consider the time-harmonic form of the scalar wave equation ($\vec{\boldsymbol{x}} = (\boldsymbol{x}, z)$)

$$(\partial_z^2 + \Delta_\perp)\hat{u} + \frac{\omega^2}{c_0^2}\big(1 + \mu(z)\big)\hat{u} = 0$$

Consider the regime $l_c \ll \lambda \ll L$, more precisely, the scaled regime

$$\omega \to \frac{\omega}{\varepsilon}, \qquad \mu(z) \to \mu\big(\frac{z}{\varepsilon^2}\big)$$

- For a point source located at $\vec{\boldsymbol{x}}_s = (\boldsymbol{x}_s, 0)$ emiting the pulse $f(t)$, a receiver located at $\vec{\boldsymbol{x}}_q = (\boldsymbol{x}_q, -L)$, the transmitted field is

$$u(t, \vec{\boldsymbol{x}}_q; \vec{\boldsymbol{x}}_s) = -\frac{1}{(2\pi)^3} \int_{\mathbb{R}} d\omega \iint_{\mathbb{R}^2} \omega^2 d\boldsymbol{\kappa}\, \hat{f}(\omega) \mathcal{G}_{\omega,\kappa} \exp\Big(-i\omega\big(t - \boldsymbol{\kappa}\cdot(\boldsymbol{x}_q - \boldsymbol{x}_s) - \frac{L}{c_0(\kappa)}\big)\Big)$$

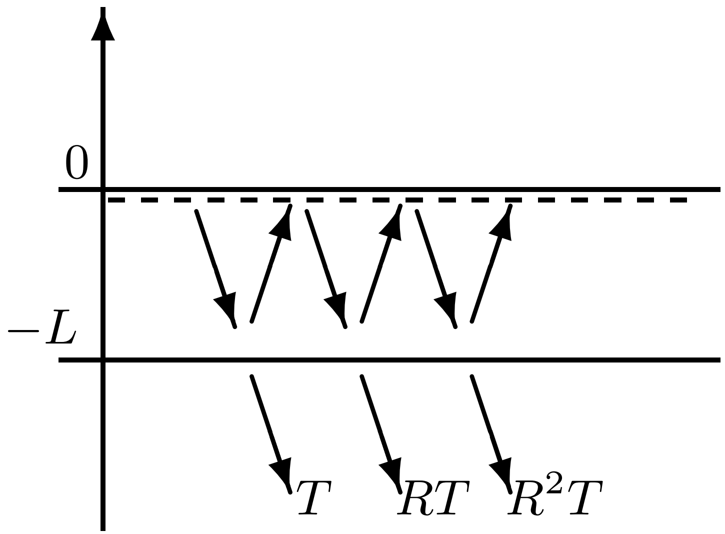
- we use a Fourier transform in time and transverse spatial coordinates.
- $c_0(\kappa)$ is the mode-dependent velocity:

$$c_0(\kappa) = \frac{c_0}{\sqrt{1 - \kappa^2 c_0^2}}$$

- $\mathcal{G}_{\omega,\kappa}$ is the random Green's function (transmission coefficient) whose moments are known [1].

[1] J.-P. Fouque, J. Garnier, G. Papanicolaou, and K. Sølna, *Wave propagation ...*, Springer, 2007.

Propagation through a randomly layered overburden: analysis (1/2)



- $\mathcal{G}_{\omega,\kappa}$ is the random Green's function for pressure release boundary conditions:

$$\mathcal{G}_{\omega,\kappa} = \sum_{j=0}^{\infty} T_{\omega,\kappa}(R_{\omega,\kappa})^j$$

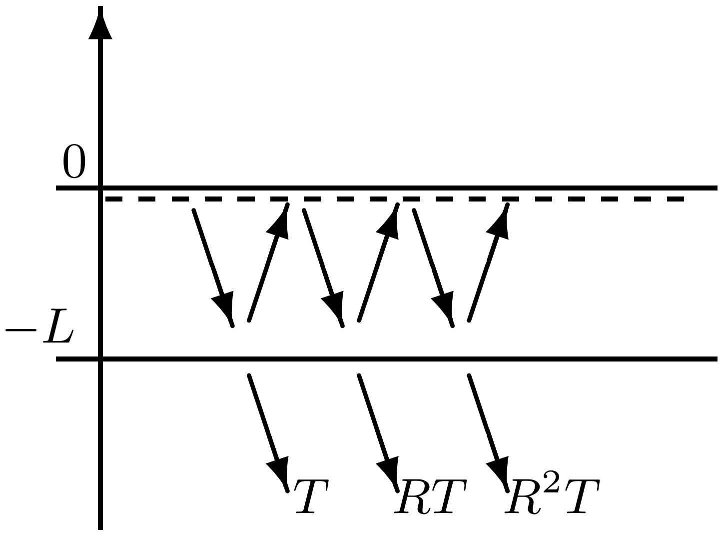
where $T_{\omega,\kappa}$ and $R_{\omega,\kappa}$ are the transmission and reflection coefficients for the random slab in $(-L, 0)$ (we have $|T_{\omega,\kappa}|^2 + |R_{\omega,\kappa}|^2 = 1$) [1].

- In a homogeneous medium $\mathcal{G}_{\omega,\kappa}$ is equal to 1 because $T_{\omega,\kappa} = 1$ and $R_{\omega,\kappa} = 0$.

[1] J.-P. Fouque, J. Garnier, G. Papanicolaou, and K. Sølna, *Wave propagation ...*, Springer, 2007.

Propagation through a randomly layered overburden: analysis (2/2)

- In a random medium:



$$\mathbb{E}\left[T_{\omega,\kappa}\right] = \exp\left(-\frac{L}{L_{\mathrm{loc}}(\omega,\kappa)}\right)$$

$$\mathbb{E}\left[|T_{\omega,\kappa}|^2\right] \sim \exp\left(-\frac{L}{4L_{\mathrm{loc}}(\omega,\kappa)}\right)$$

$$\mathcal{G}_{\omega,\kappa} = \sum_{j=0}^{\infty} T_{\omega,\kappa}(R_{\omega,\kappa})^j$$

$$\mathbb{E}\left[|\mathcal{G}_{\omega,\kappa}|^2\right] = 1$$

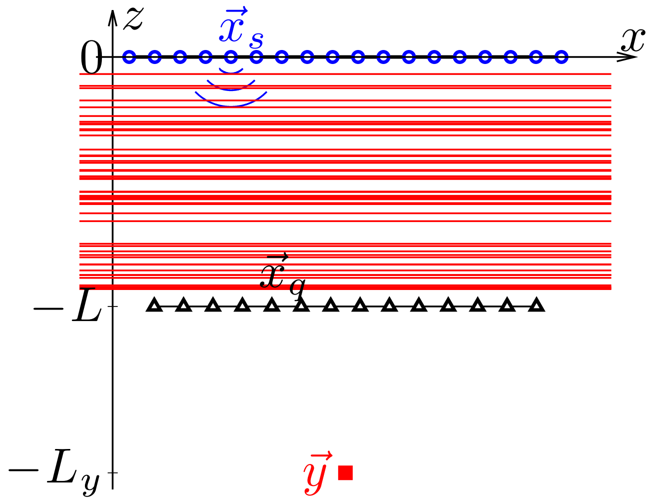
- $\mathbb{E}[T_{\omega,\kappa}] \ll 1$, i.e. most of the energy is in the incoherent fluctuations $\Longrightarrow$ Coherent imaging methods fail.

- Exponential decay of $\mathbb{E}[|T_{\omega,\kappa}|^2]$ specific to randomly layered media $\Longrightarrow$ Transmitted signals are very long.

- $\mathbb{E}\left[|\mathcal{G}_{\omega,\kappa}|^2\right] = 1 \Longrightarrow$ Good (but incoherent) illumination.

- The second-order moment $\mathbb{E}[\mathcal{G}_{\omega,\kappa}\overline{\mathcal{G}_{\omega,\kappa'}}]$ is given in terms of a a transport-type equation.

[1] J.-P. Fouque, J. Garnier, G. Papanicolaou, and K. Sølna, *Wave propagation ...*, Springer, 2007.



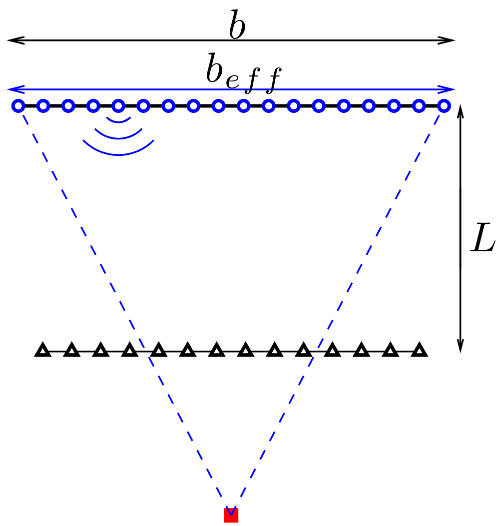
- Assume that:

- the source aperture is b and the receiver aperture is a (use continuum approximation for the source and receiver arrays).
- there is a point reflector at $\vec{\boldsymbol{y}} = (\boldsymbol{y}, -L_y)$ (use the Born approximation for the reflector).
- the localization length $L_{\rm loc}$ is smaller than L (strong scattering):

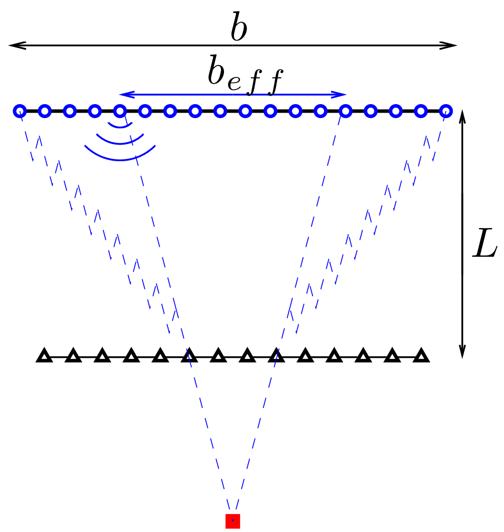
$$L_{\rm loc} = \frac{4c_0^2}{\gamma\omega_0^2}, \qquad \gamma = \int_{-\infty}^{\infty} \mathbb{E}[\mu(0)\mu(z)]dz$$

- There are two critical lengths:

$$a_{\rm eff} = b_{\rm eff}\frac{L_y - L}{L_y}, \qquad b_{\rm eff}^2 = 4L_{\rm loc}L$$



Homogeneous medium

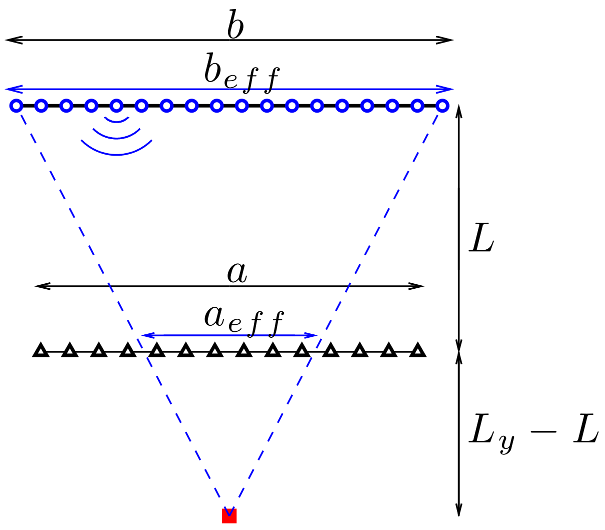


Randomly layered medium

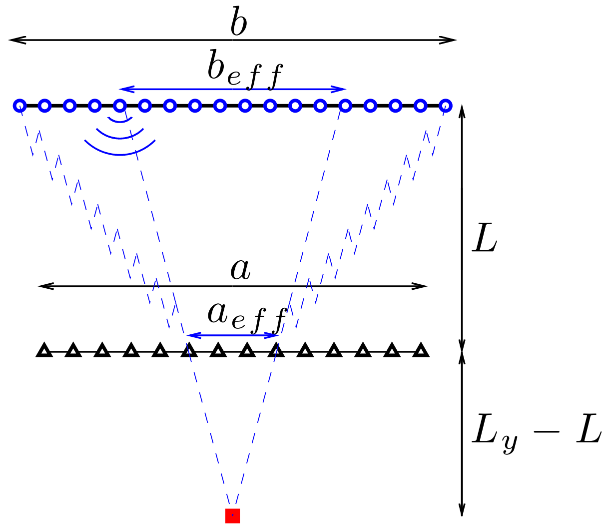
Effective source aperture:

$b_{\rm eff} = b$

$b_{\rm eff}^2 = 4 L_{\rm loc} L \ \ (\ll b^2)$



Homogeneous medium



Randomly layered medium

Effective source aperture:

$$b_{\text{eff}} = b \qquad\qquad b_{\text{eff}}^2 = 4 L_{\text{loc}} L$$

Effective receiver aperture:

$$a_{\text{eff}} = b \frac{L_y - L}{L_y} \qquad\qquad a_{\text{eff}} = b_{\text{eff}} \frac{L_y - L}{L_y}$$

Migration of the cross correlation matrix

- The Kirchhoff Migration function for the search point $\vec{\boldsymbol{y}}^S$ is

$$\mathcal{I}(\vec{\boldsymbol{y}}^S) = \frac{1}{N_{\rm q}^2} \sum_{q,q'=1}^{N_{\rm q}} \mathcal{C}\Big(\frac{|\vec{\boldsymbol{x}}_q - \vec{\boldsymbol{y}}^S| + |\vec{\boldsymbol{y}}^S - \vec{\boldsymbol{x}}_{q'}|}{c_0}, \vec{\boldsymbol{x}}_q, \vec{\boldsymbol{x}}_{q'} \Big)$$

- The imaging function is statistically stable ($\lambda \ll b, L$).

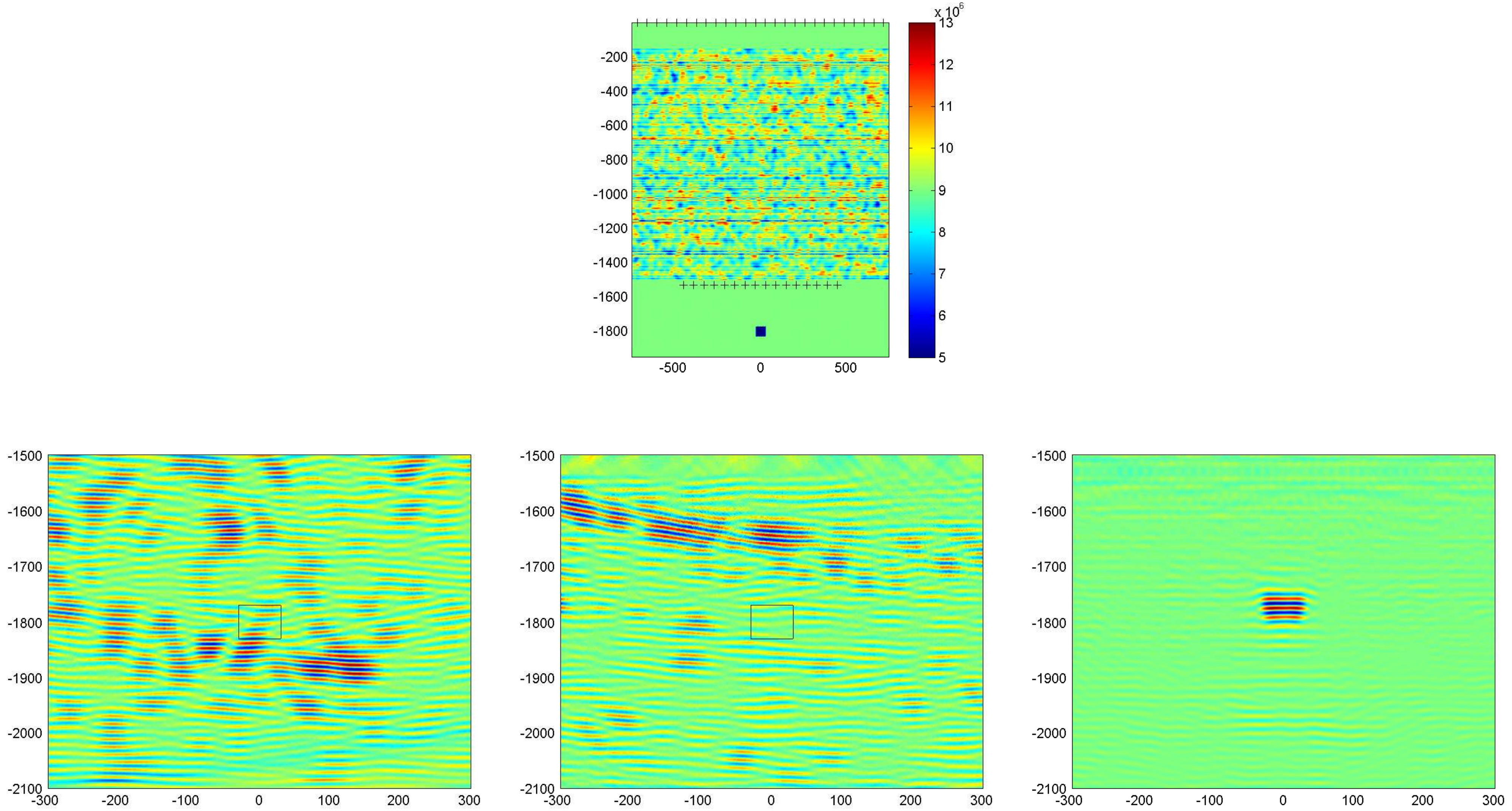
- The cross range resolution is $\dfrac{\lambda_0(L_y - L)}{a_{\rm eff}}$.

The range resolution is $\dfrac{c_0}{B}\big(1 + \dfrac{B^2 L}{4\omega_0^2 L_{\rm loc}}\big)^{1/2}$.

- Since $a_{\rm eff}\, |_{\rm rand} < a_{\rm eff}\, |_{\rm homo}$, this shows that scattering does not help (it reduces the angular diversity of the illumination) !

Numerical simulations in a strongly scattering medium

Numerical simulations



Top: computational setup.

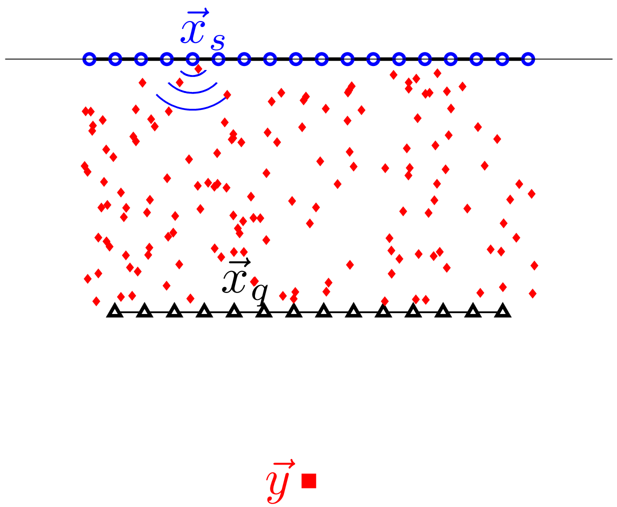
Left: image obtained with Kirchhoff Migration using the surface array.

Middle: image obtained with Kirchhoff Migration using the bottom array.

Right: image obtained with the cross correlation technique using the bottom array.

Conclusions

• Ideal situation for the cross correlation technique (with active sources everywhere):



• What is the role of scattering if the sources are spatially localized ?
The answer depends on the scattering regime:
- in the isotropic case, random scattering helps (enhances the source aperture).
- in the layered case, random scattering is bad (reduces the source aperture).

• Same conclusion for the C^3 technique.

• Here the medium was assumed to be homogeneous in the underburden (between the secondary array and the reflector).
What happens if it is scattering ? Modify the cut-off parameters of the CINT functional (for weakly scattering underburden).

Perspectives

- Space surveillance and imaging with airborne passive synthetic aperture arrays.