# High-order statistics for the random paraxial wave equation. Application to correlation-based imaging

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## Wave propagation in random media

- Wave equation:

$$\frac{1}{c^2(\vec{\boldsymbol{x}})}\frac{\partial^2 u}{\partial t^2}(t,\vec{\boldsymbol{x}}) - \Delta_{\vec{\boldsymbol{x}}} u(t,\vec{\boldsymbol{x}}) = F(t,\vec{\boldsymbol{x}})$$

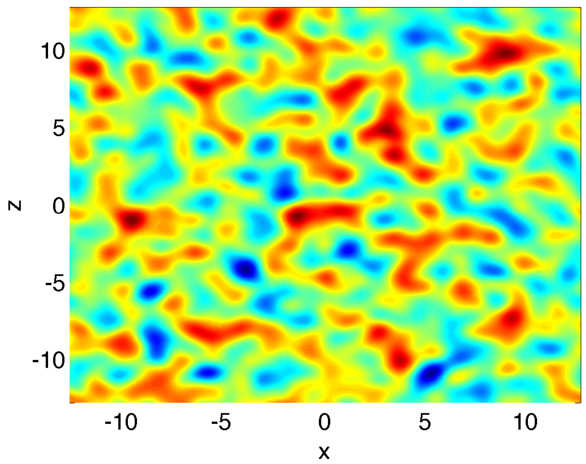
- Time-harmonic source in the plane $z = 0$: $F(t,\vec{\boldsymbol{x}}) = \delta(z) f(\boldsymbol{x}) e^{-i\omega t}$ (with $\vec{\boldsymbol{x}} = (\boldsymbol{x}, z)$).

- Random medium model:

$$\frac{1}{c^2(\vec{\boldsymbol{x}})} = \frac{1}{c_o^2}\big(1 + \mu(\vec{\boldsymbol{x}})\big)$$

$c_o$ is a reference speed,

$\mu(\vec{\boldsymbol{x}})$ is a zero-mean random process.



# Wave propagation in the random paraxial regime

• Consider the time-harmonic wave equation (with $\vec{\boldsymbol{x}} = (\boldsymbol{x}, z)$ and $\Delta = \Delta_\perp + \partial_z^2$)

$$(\partial_z^2 + \Delta_\perp)\hat{u} + \frac{\omega^2}{c_o^2}\big(1 + \mu(\boldsymbol{x}, z)\big)\hat{u} = -\delta(z) f(\boldsymbol{x}).$$

The function $\hat{\phi}$ (slowly-varying envelope of a plane wave) defined by

$$\hat{u}(\omega, \boldsymbol{x}, z) = \frac{i c_o}{2\omega} e^{i\frac{\omega z}{c_o}} \hat{\phi}\big(\omega, \boldsymbol{x}, z\big)$$

satisfies

$$\partial_z^2 \hat{\phi} + \left(2i\frac{\omega}{c_o}\partial_z \hat{\phi} + \Delta_\perp \hat{\phi} + \frac{\omega^2}{c_o^2}\mu\big(\boldsymbol{x}, z\big)\hat{\phi}\right) = 2i\frac{\omega}{c_o}\delta(z) f(\boldsymbol{x}).$$

• In the paraxial regime "$\lambda \ll l_c, r_o \ll L$", the forward-scattering approximation in direction $z$ is valid and $\hat{\phi}$ satisfies the Itô-Schrödinger equation [1]

$$d_z \hat{\phi} = \frac{i c_o}{2\omega}\Delta_\perp \hat{\phi} dz + \frac{i\omega}{2c_o}\hat{\phi} \circ dB(\boldsymbol{x}, z), \qquad \hat{\phi}(z = 0, \boldsymbol{x}) = f(\boldsymbol{x})$$

with $B(\boldsymbol{x}, z)$ Brownian field $\mathbb{E}[B(\boldsymbol{x}, z)B(\boldsymbol{x}', z')] = \gamma(\boldsymbol{x} - \boldsymbol{x}') \min(z, z')$ and

$$\gamma(\boldsymbol{x}) = \int_{-\infty}^{\infty} \mathbb{E}[\mu(\boldsymbol{0}, 0)\mu(\boldsymbol{x}, z)]dz$$

[1] J. Garnier and K. Sølna, *Ann. Appl. Probab.* **19**, 318 (2009).

# Moment calculations in the random paraxial regime

Consider

$$d_z\hat{\phi} = \frac{ic_o}{2\omega}\Delta_\perp\hat{\phi}dz + \frac{i\omega}{2c_o}\hat{\phi}\circ dB(\boldsymbol{x},z)$$

starting from $\hat{\phi}(\boldsymbol{x}, z=0) = f(\boldsymbol{x})$.

• By Itô's formula,

$$\frac{d}{dz}\mathbb{E}[\hat{\phi}] = \frac{ic_o}{2\omega}\Delta_\perp\mathbb{E}[\hat{\phi}] - \frac{\omega^2\gamma(\mathbf{0})}{8c_o^2}\mathbb{E}[\hat{\phi}]$$

and therefore

$$\mathbb{E}\big[\hat{\phi}(\boldsymbol{x},z)\big] = \hat{\phi}_0(\boldsymbol{x},z)\exp\Big(-\frac{\gamma(\mathbf{0})\omega^2 z}{8c_o^2}\Big),$$

where $\gamma(\boldsymbol{x}) = \int_{-\infty}^{\infty}\mathbb{E}[\mu(\mathbf{0},0)\mu(\boldsymbol{x},z)]dz$ and $\hat{\phi}_0$ is the solution in the homogeneous medium.

• Strong damping of the coherent wave.

$\Longrightarrow$ Identification of the *scattering mean free path* $Z_{\rm sca} = \frac{8c_o^2}{\gamma(\mathbf{0})\omega^2}$.

$\Longrightarrow$ Coherent imaging methods (such as Kirchhoff migration, Reverse-Time migration) fail.

## Moment calculations in the random paraxial regime

• The mean Wigner transform defined by

$$W(\boldsymbol{x},\boldsymbol{\xi},z) = \int_{\mathbb{R}^2} \exp\big(-i\boldsymbol{\xi}\cdot\boldsymbol{y}\big)\mathbb{E}\Big[\hat{\phi}\big(\boldsymbol{x}+\frac{\boldsymbol{y}}{2},z\big)\overline{\hat{\phi}}\big(\boldsymbol{x}-\frac{\boldsymbol{y}}{2},z\big)\Big]\,d\boldsymbol{y},$$

is the angularly-resolved mean wave energy density. By Itô's formula, it solves a *radiative transport-like equation*

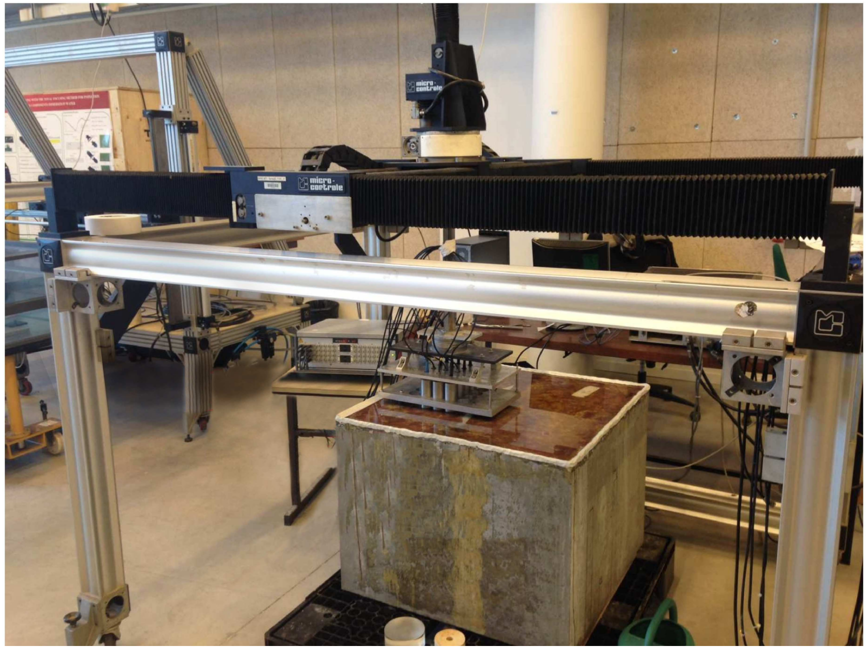
$$\frac{\partial W}{\partial z} + \frac{c_o}{\omega}\boldsymbol{\xi}\cdot\nabla_{\boldsymbol{x}}W = \frac{\omega^2}{4(2\pi)^2c_o^2}\int_{\mathbb{R}^2}\hat{\gamma}(\boldsymbol{\kappa})\Big[W(\boldsymbol{\xi}-\boldsymbol{\kappa})-W(\boldsymbol{\xi})\Big]d\boldsymbol{\kappa},$$

starting from $W(\boldsymbol{x},\boldsymbol{\xi},z=0)=W_0(\boldsymbol{x},\boldsymbol{\xi})$, the Wigner transform of the initial field $f$.

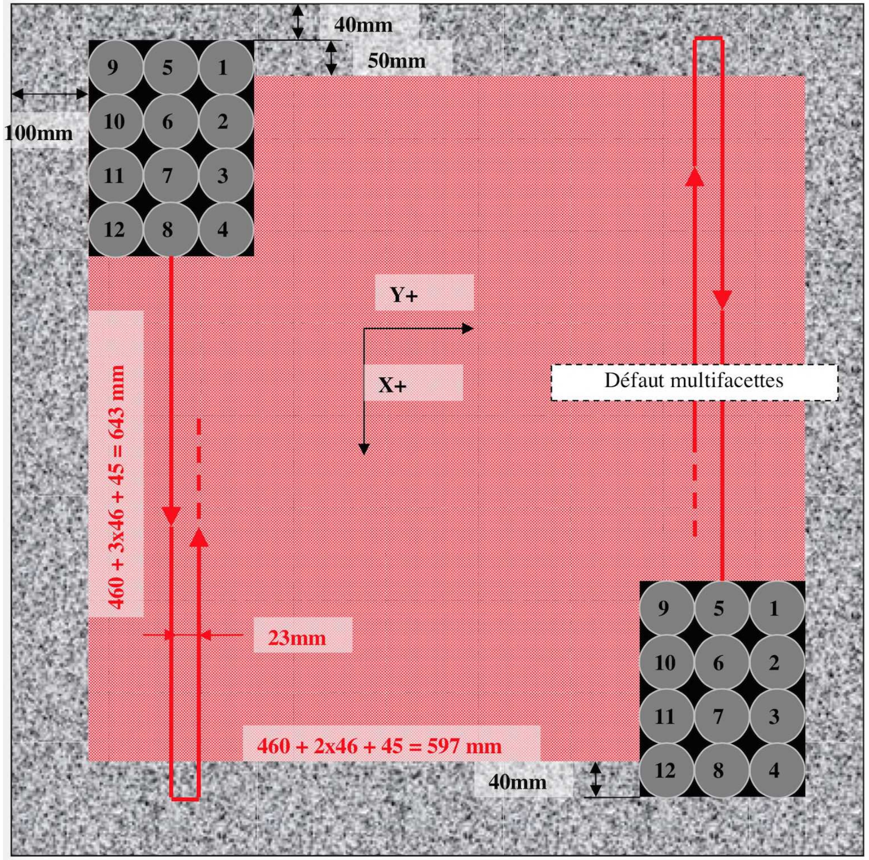
• The fields at nearby points are correlated and their correlations contain information about the medium.

$\Longrightarrow$ One should use (migrate) cross correlations for imaging in random media.

# Application: Ultrasound echography in concrete



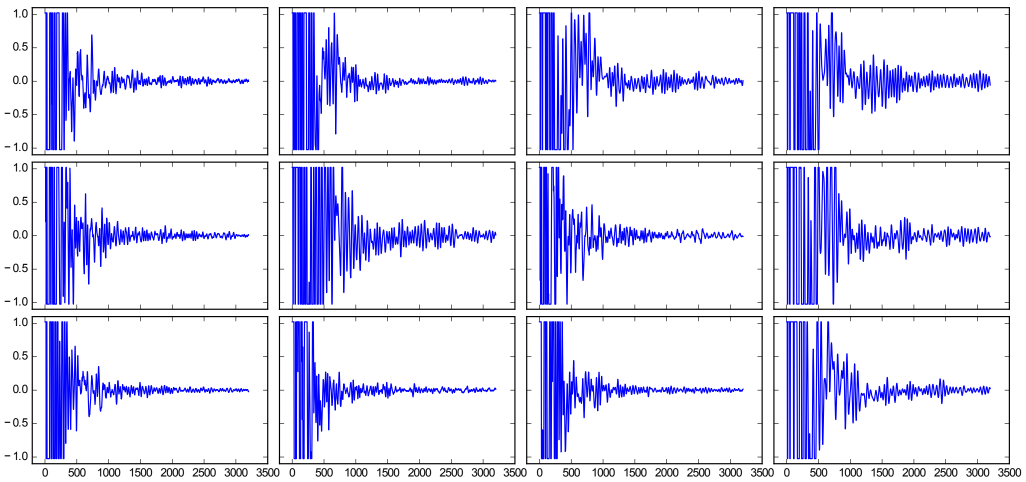
Experimental set-up




Acquisition geometry (top view)

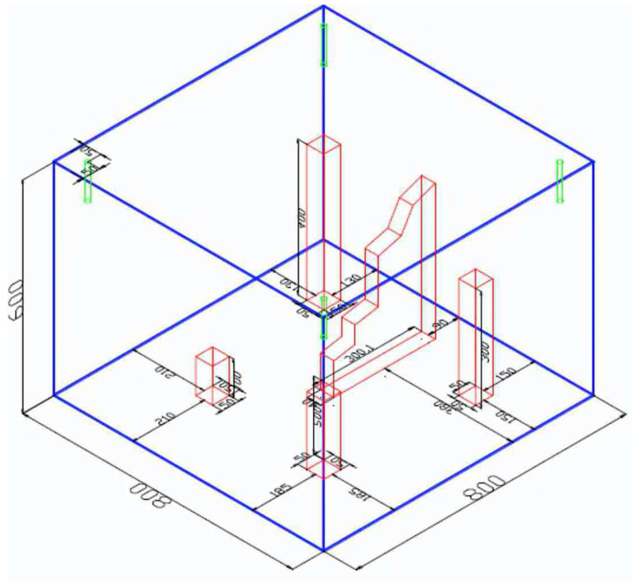
Concrete: highly scattering medium for ultrasonic waves.

# Application: Ultrasound echography in concrete




Data

$\overset{?}{\longrightarrow}$

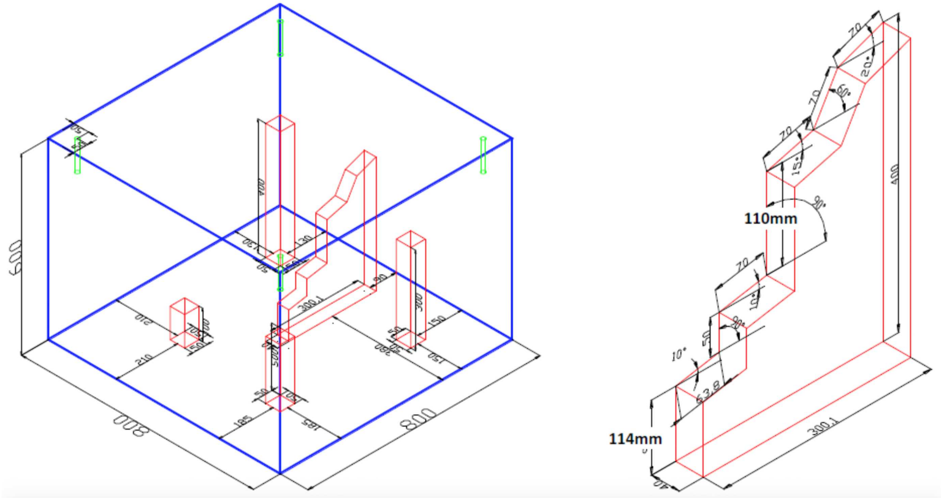


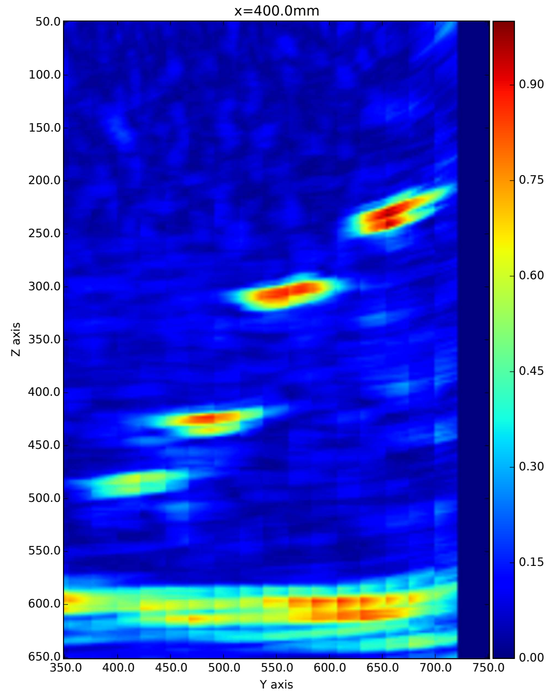

Real configuration

The recorded signals are very “noisy” due to scattering.

$\hookrightarrow$ Standard imaging techniques fail.

# Application: Ultrasound echography in concrete




Real configuration

Image (2D slice)

Image obtained by travel-time migration of *well-chosen* cross correlations of data.

# Moment calculations in the random paraxial regime

- Consider

$$d\hat{\phi} = \frac{ic_o}{2\omega}\Delta_\perp \hat{\phi} dz + \frac{i\omega}{2c_o}\hat{\phi} \circ dB(\boldsymbol{x}, z)$$

starting from $\hat{\phi}(\boldsymbol{x}, z = 0) = f(\boldsymbol{x})$.

- Let us consider the fourth-order moment:

$$\begin{aligned} M_4(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{q}_1, \boldsymbol{q}_2, z) &= \mathbb{E}\Big[\hat{\phi}\big(\frac{\boldsymbol{r}_1 + \boldsymbol{r}_2 + \boldsymbol{q}_1 + \boldsymbol{q}_2}{2}, z\big)\hat{\phi}\big(\frac{\boldsymbol{r}_1 - \boldsymbol{r}_2 + \boldsymbol{q}_1 - \boldsymbol{q}_2}{2}, z\big) \\ &\quad \times \overline{\hat{\phi}}\big(\frac{\boldsymbol{r}_1 + \boldsymbol{r}_2 - \boldsymbol{q}_1 - \boldsymbol{q}_2}{2}, z\big)\overline{\hat{\phi}}\big(\frac{\boldsymbol{r}_1 - \boldsymbol{r}_2 - \boldsymbol{q}_1 + \boldsymbol{q}_2}{2}, z\big)\Big] \end{aligned}$$

By Itô's formula,

$$\frac{\partial M_4}{\partial z} = \frac{ic_o}{\omega}\big(\nabla_{\boldsymbol{r}_1} \cdot \nabla_{\boldsymbol{q}_1} + \nabla_{\boldsymbol{r}_2} \cdot \nabla_{\boldsymbol{q}_2}\big)M_4 + \frac{\omega^2}{4c_o^2}U_4(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{r}_1, \boldsymbol{r}_2)M_4,$$

with the generalized potential

$$\begin{aligned} U_4(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{r}_1, \boldsymbol{r}_2) &= \gamma(\boldsymbol{q}_2 + \boldsymbol{q}_1) + \gamma(\boldsymbol{q}_2 - \boldsymbol{q}_1) + \gamma(\boldsymbol{r}_2 + \boldsymbol{q}_1) + \gamma(\boldsymbol{r}_2 - \boldsymbol{q}_1) \\ &\quad - \gamma(\boldsymbol{q}_2 + \boldsymbol{r}_2) - \gamma(\boldsymbol{q}_2 - \boldsymbol{r}_2) - 2\gamma(\boldsymbol{0}). \end{aligned}$$

These moment equations have been known and studied for a long time, in particular to prove the Gaussian conjecture [1].

[1] A. Ishimaru, *Wave Propagation and Scattering in Random Media*, Academic Press, San Diego, 1978.

# Moment calculations in the random paraxial regime

Take Fourier transform:

$$\hat{M}_4(\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2,z) = \iiiint M_4(\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{r}_1,\boldsymbol{r}_2,z) \times \exp\big(-i\boldsymbol{q}_1\cdot\boldsymbol{\xi}_1 - i\boldsymbol{r}_1\cdot\boldsymbol{\zeta}_1 - i\boldsymbol{q}_2\cdot\boldsymbol{\xi}_2 - i\boldsymbol{r}_2\cdot\boldsymbol{\zeta}_2\big)d\boldsymbol{r}_1 d\boldsymbol{r}_2 d\boldsymbol{q}_1 d\boldsymbol{q}_2.$$

- In the regime "$\lambda \ll l_c \ll r_o \ll L$" [1]

$$\hat{M}_4(\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2,z) \simeq \Phi(K,A,f)(\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\zeta}_1,\boldsymbol{\zeta}_2,z)$$

where

$$K(z) = (2\pi)^8 \exp\Big(-\frac{\omega^2}{2c_o^2}\gamma(\boldsymbol{0})z\Big),$$

$$A(\boldsymbol{\xi},\boldsymbol{\zeta},z) = \frac{1}{2(2\pi)^2}\int \Big[\exp\Big(\frac{\omega^2}{4c_o^2}\int_0^z \gamma\big(\boldsymbol{x}+\frac{c_o\boldsymbol{\zeta}}{\omega}z'\big)dz'\Big) - 1\Big]\exp\big(-i\boldsymbol{\xi}\cdot\boldsymbol{x}\big)d\boldsymbol{x}.$$

[1] J. Garnier and K. Sølna, *ARMA* **220** (2016) 37.

## Scintillation

Assume that $f(\boldsymbol{x}) = \exp\big(-\frac{|\boldsymbol{x}|^2}{2r_o^2}\big)$.

- The scintillation index defined as:

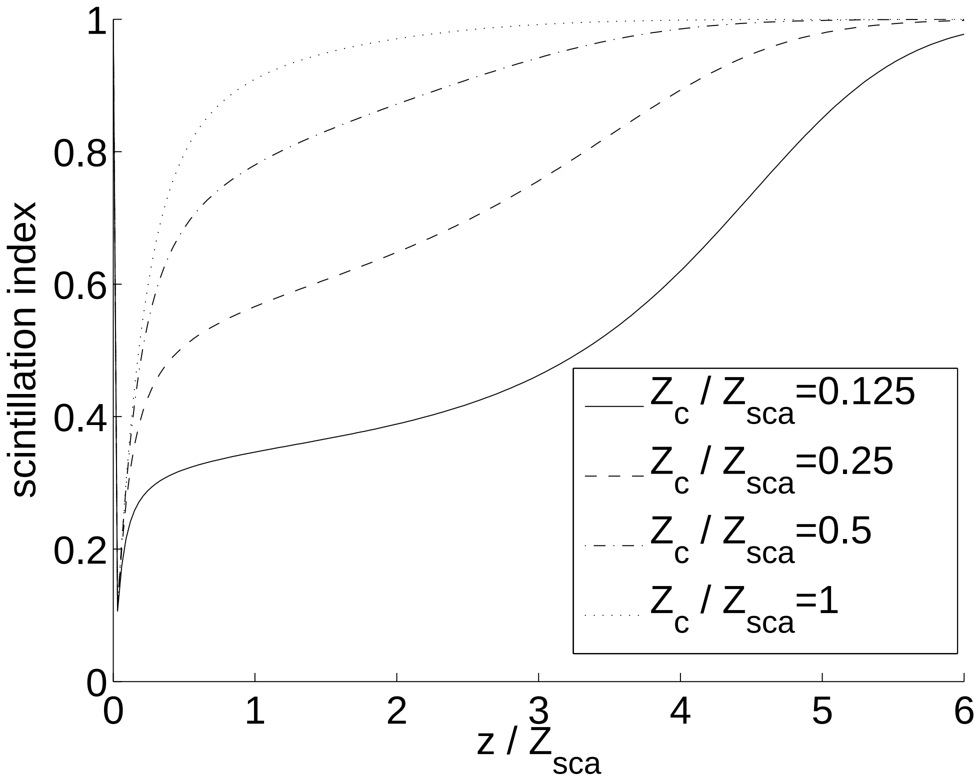
$$S(\boldsymbol{x},z) := \frac{\mathbb{E}\Big[\big|\hat{\phi}(\boldsymbol{x},z)\big|^4\Big] - \mathbb{E}\Big[\big|\hat{\phi}(\boldsymbol{x},z)\big|^2\Big]^2}{\mathbb{E}\Big[\big|\hat{\phi}(\boldsymbol{x},z)\big|^2\Big]^2}$$

satisfies:

$$S(\boldsymbol{x},z) = 1 - \frac{1}{\Big|\frac{1}{4\pi}\int_{\mathbb{R}^2}\exp\Big(\frac{\omega^2}{4c_o^2}\int_0^z \gamma\Big(\boldsymbol{u}\frac{c_o z'}{\omega r_o}\Big)dz' - \frac{|\boldsymbol{u}|^2}{4} + i\boldsymbol{u}\cdot\frac{\boldsymbol{x}}{r_o} + \frac{|\boldsymbol{x}|^2}{r_o^2}\Big)d\boldsymbol{u}\Big|^2}.$$

The physical conjecture is that $S \simeq 1$ when the propagation distance is larger than the scattering mean free path, as it should be for a (complex) Gaussian process.

# Scintillation




Scintillation index at the beam center $S(z, \mathbf{0})$ as a function of the propagation distance for different values of $Z_{\rm sca} = \frac{8c_o^2}{\omega^2\gamma(\mathbf{0})}$ and $Z_{\rm c} = \frac{\omega r_o \ell_{\rm c}}{c_o}$. Here $\gamma(\boldsymbol{x}) = \gamma(\mathbf{0})\exp(-|\boldsymbol{x}|^2/\ell_{\rm c}^2)$.

## Stability of the Wigner transform of the field

$$W(\boldsymbol{r},\boldsymbol{\xi},z) := \int_{\mathbb{R}^2} \exp\big(-i\boldsymbol{\xi}\cdot\boldsymbol{q}\big)\hat{\phi}\big(\boldsymbol{r}+\frac{\boldsymbol{q}}{2},z\big)\overline{\hat{\phi}}\big(\boldsymbol{r}-\frac{\boldsymbol{q}}{2},z\big)d\boldsymbol{q}.$$

Let us consider two positive parameters $r_{\rm s}$ and $\xi_{\rm s}$ and define the smoothed Wigner transform:

$$W_{\rm s}(\boldsymbol{r},\boldsymbol{\xi},z) = \frac{1}{(2\pi)^2 r_{\rm s}^2\xi_{\rm s}^2}\iint_{\mathbb{R}^2\times\mathbb{R}^2} W(\boldsymbol{r}-\boldsymbol{r}',\boldsymbol{\xi}-\boldsymbol{\xi}',z)\exp\Big(-\frac{|\boldsymbol{r}'|^2}{2r_{\rm s}^2}-\frac{|\boldsymbol{\xi}'|^2}{2\xi_{\rm s}^2}\Big)d\boldsymbol{r}'d\boldsymbol{\xi}'.$$

- The coefficient of variation $C_{\rm s}$ of the smoothed Wigner transform defined by:

$$C_{\rm s}(\boldsymbol{r},\boldsymbol{\xi},z) := \frac{\sqrt{\mathbb{E}[W_{\rm s}(\boldsymbol{r},\boldsymbol{\xi},z)^2]-\mathbb{E}[W_{\rm s}(\boldsymbol{r},\boldsymbol{\xi},z)]^2}}{\mathbb{E}[W_{\rm s}(\boldsymbol{r},\boldsymbol{\xi},z)]}.$$

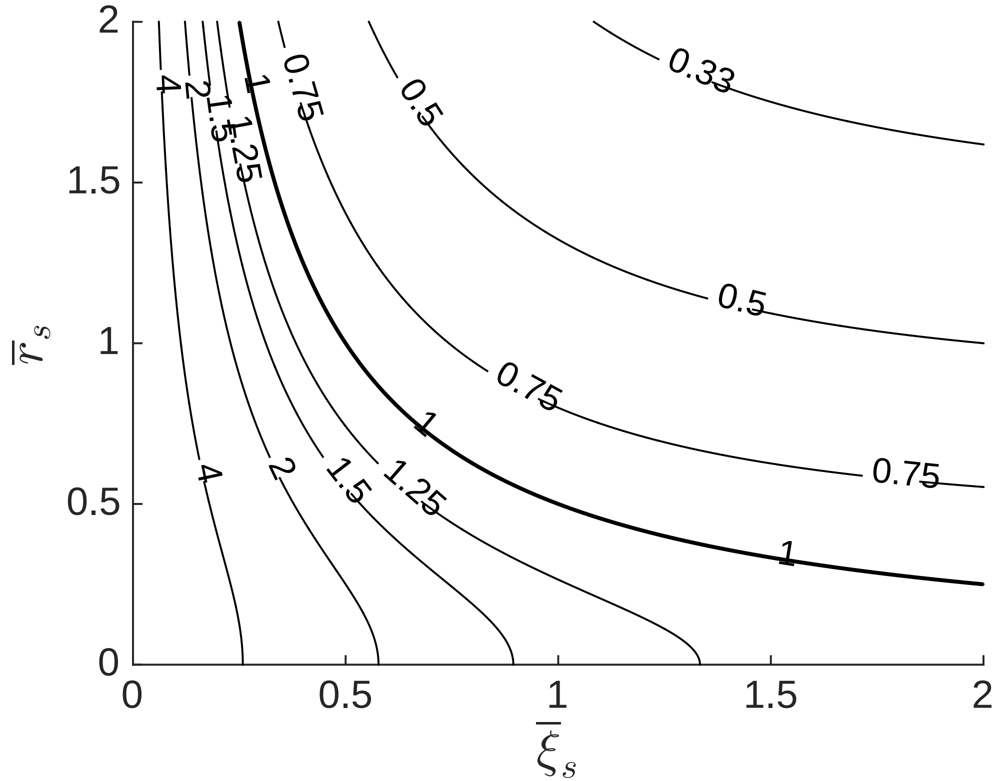
satisfies

$$C_{\rm s}(\boldsymbol{r},\boldsymbol{\xi},z) \simeq \left(\frac{\frac{1}{\xi_{\rm s}^2\rho_z^2}+1}{\frac{4r_{\rm s}^2}{\rho_z^2}+1}\right)^{1/2}, \qquad \rho_z^2 = \frac{\ell_{\rm c}^2}{4Z_{\rm sca}z}\,\frac{r_o^2+\frac{8c_o^2z^3}{3\omega^2\ell_{\rm c}^2Z_{\rm sca}}}{r_o^2+\frac{2c_o^2z^3}{3\omega^2\ell_{\rm c}^2Z_{\rm sca}}},$$

when

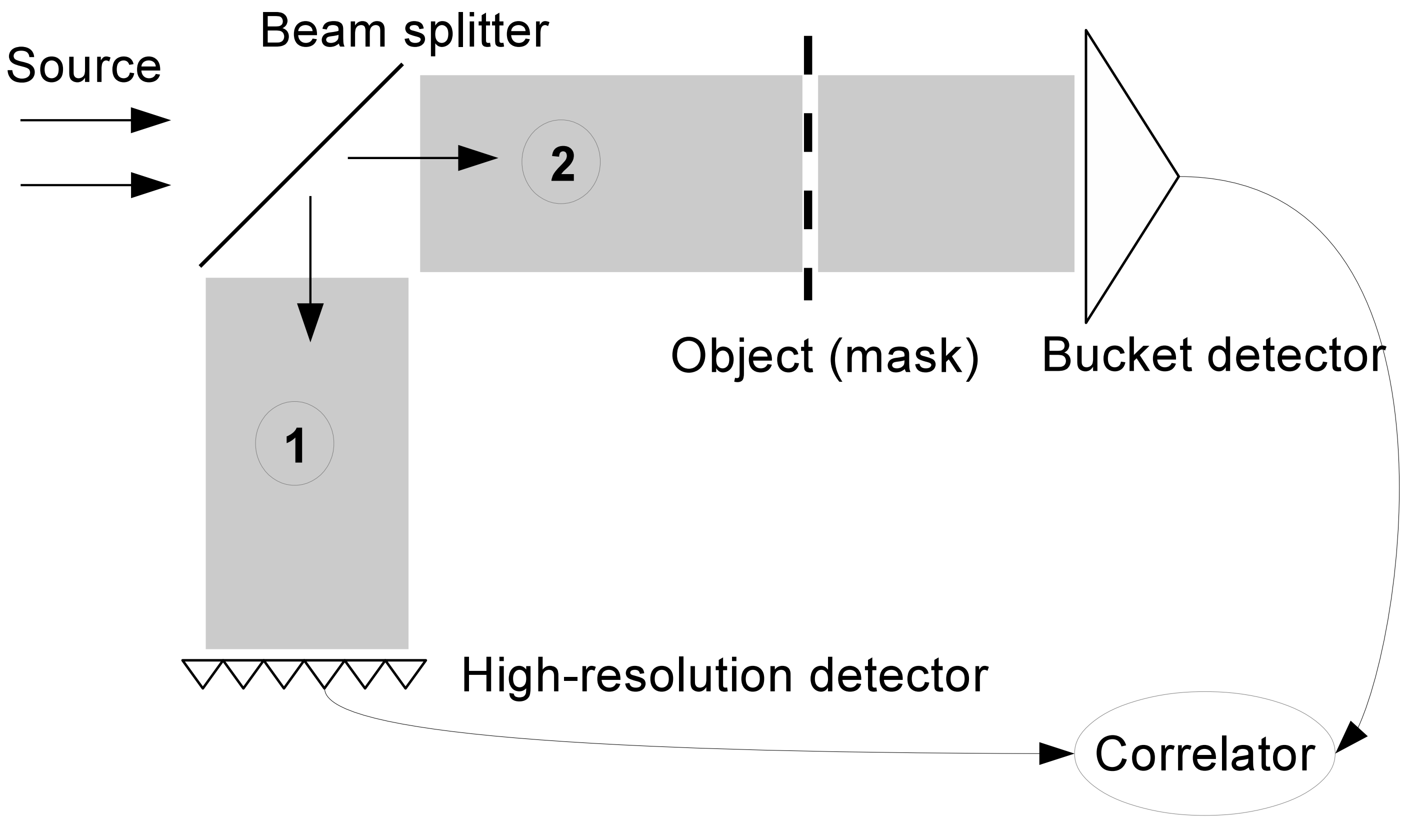
$$\gamma(\boldsymbol{x}) = \gamma(\boldsymbol{0})\Big[1-\frac{|\boldsymbol{x}|^2}{\ell_{\rm c}^2}+o\Big(\frac{|\boldsymbol{x}|^2}{\ell_{\rm c}^2}\Big)\Big], \qquad z \gg Z_{\rm sca} = \frac{8c_o^2}{\gamma(\boldsymbol{0})\omega^2}.$$

# Stability of the Wigner transform of the field




Contour levels of the coefficient of variation of the smoothed Wigner transform. Here $\overline{r}_{\rm s} = r_{\rm s}/\rho_z$ and $\overline{\xi}_{\rm s} = \xi_{\rm s}\rho_z$.

# Ghost imaging




- Noise source (laser light passed through a rotating glass diffuser).
- without object in path 1; a high-resolution detector measures the spatially-resolved intensity $I_1(t, \boldsymbol{x})$.
- with object (mask) in path 2; a single-pixel detector measures the spatially-integrated intensity $I_2(t)$.

Experimental result: the correlation of $I_1(\cdot, \boldsymbol{x})$ and $I_2(\cdot)$ is an image of the object [1,2].

[1] A. Valencia et al., *PRL* **94**, 063601 (2005); [2] J. H. Shapiro et al., *Quantum Inf. Process* **1**, 949 (2012).

# Ghost imaging

- Wave equation in paths 1 and 2:

$$\frac{1}{c_j^2(\vec{\boldsymbol{x}})}\frac{\partial^2 u_j}{\partial t^2} - \Delta_{\vec{\boldsymbol{x}}} u_j = e^{-i\omega_o t} n(t,\boldsymbol{x})\delta(z) + c.c., \qquad \vec{\boldsymbol{x}} = (\boldsymbol{x}, z) \in \mathbb{R}^2 \times \mathbb{R}, \qquad j = 1,2$$

- Noise source (with Gaussian statistics):

$$\left\langle n(t,\boldsymbol{x})\overline{n(t,\boldsymbol{x}')}\right\rangle = F(t-t')\exp\Big(-\frac{|\boldsymbol{x}|^2}{r_o^2}\Big)\delta(\boldsymbol{x}-\boldsymbol{x}')$$

with the width of $\hat{F}(\omega)$ much smaller than $\omega_o$.

- Wave fields:

$$u_j(t,\vec{\boldsymbol{x}}) = v_j(t,\vec{\boldsymbol{x}})e^{-i\omega_o t} + c.c., \qquad j = 1,2$$

- Intensity measurements:

$$\begin{aligned} I_1(t,\boldsymbol{x}) &= |v_1(t,(\boldsymbol{x},L))|^2 \text{ in the plane of the high-resolution detector} \\ I_2(t) &= \int_{\mathbb{R}^2} |v_2(t,(\boldsymbol{x}',L+L_0))|^2 d\boldsymbol{x}' \text{ in the plane of the bucket detector} \end{aligned}$$

- Correlation:

$$C_T(\boldsymbol{x}) = \frac{1}{T}\int_0^T I_1(t,\boldsymbol{x})I_2(t)dt - \Big(\frac{1}{T}\int_0^T I_1(t,\boldsymbol{x})dt\Big)\Big(\frac{1}{T}\int_0^T I_2(t)dt\Big)$$

## Ghost imaging in homogeneous media

- Resolution analysis in homogeneous media.
- Model for the object: Mask $\mathcal{T}(\boldsymbol{x})$ in the plane $z = L$.
- Result:

$$C_T(\boldsymbol{x}) \stackrel{T\to\infty}{\longrightarrow} C^{(1)}(\boldsymbol{x}) = \int_{\mathbb{R}^2} h(\boldsymbol{x}-\boldsymbol{z})|\mathcal{T}(\boldsymbol{z})|^2 d\boldsymbol{z}$$

with

$$h(\boldsymbol{x}) = \frac{r_o^4}{2^8\pi^2 L^2}\exp\Big(-\frac{|\boldsymbol{x}|^2}{4\rho_{\rm gi0}^2}\Big), \qquad \rho_{\rm gi0}^2 = \frac{c_o^2 L^2}{2\omega_o^2 r_o^2}$$

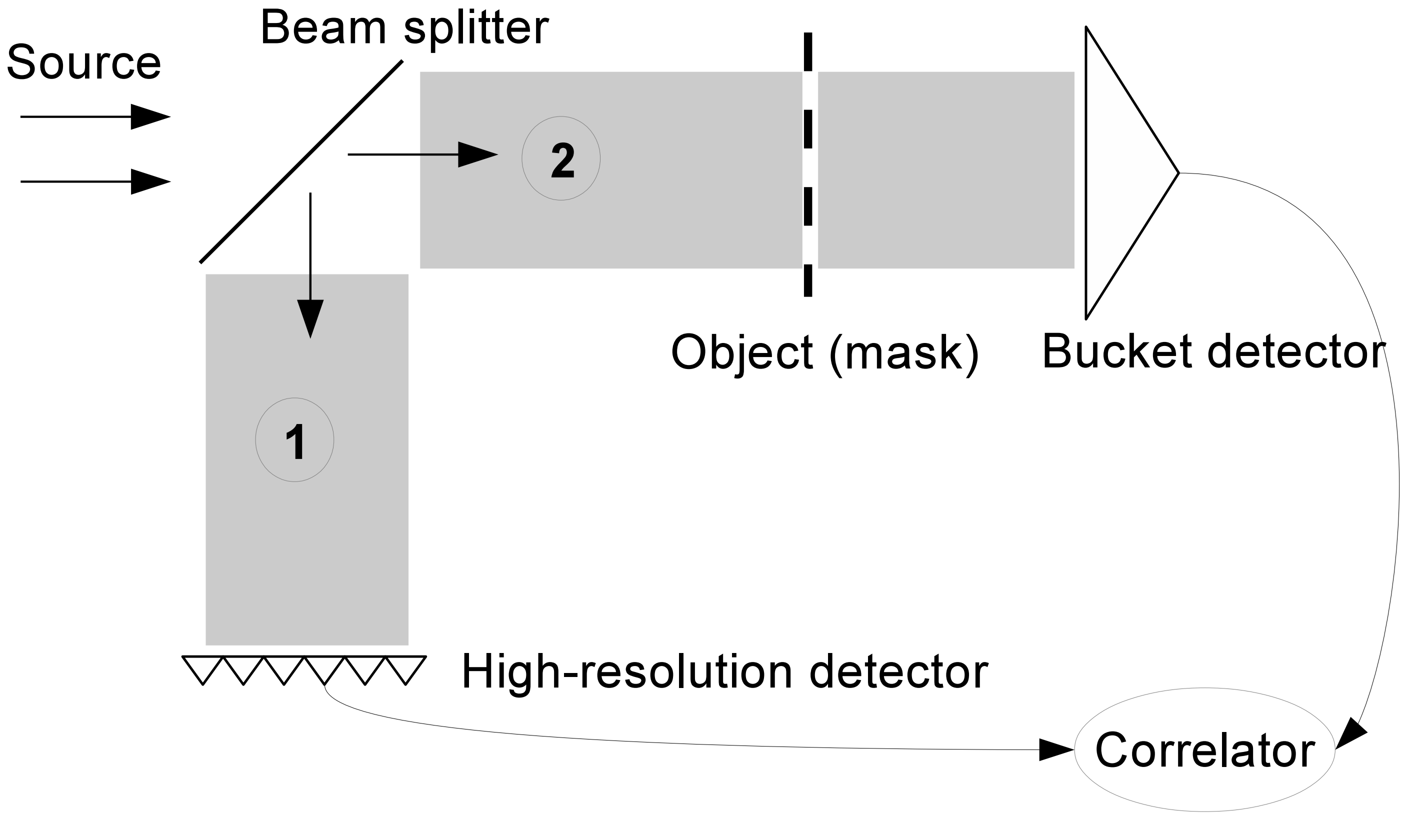
Resolution: $\rho_{\rm gi0} \sim \lambda_o L/r_o$ (Rayleigh resolution formula).

**Sketch of ideal proof.** Use the Gaussian summation rule (the fourth-order moments of Gaussian random fields can be expressed in terms of sums of products of second-order moments).

If $v(\boldsymbol{x})$ is a complex symmetric circular Gaussian random field, then

$$\text{Cov}\big(|v(\boldsymbol{x})|^2, |v(\boldsymbol{x}')|^2\big) = \big|\text{Cov}\big(v(\boldsymbol{x}), \overline{v(\boldsymbol{x}')}\big)\big|^2$$

# Ghost imaging in heterogeneous media




The medium in paths 1 and 2 is heterogeneous (for instance, turbulent atmosphere).
They are two independent realizations with the same distribution.

# Ghost imaging in heterogeneous media

- Resolution analysis in randomly heterogeneous media.
- If the propagation distance is larger than the scattering mean free path, then

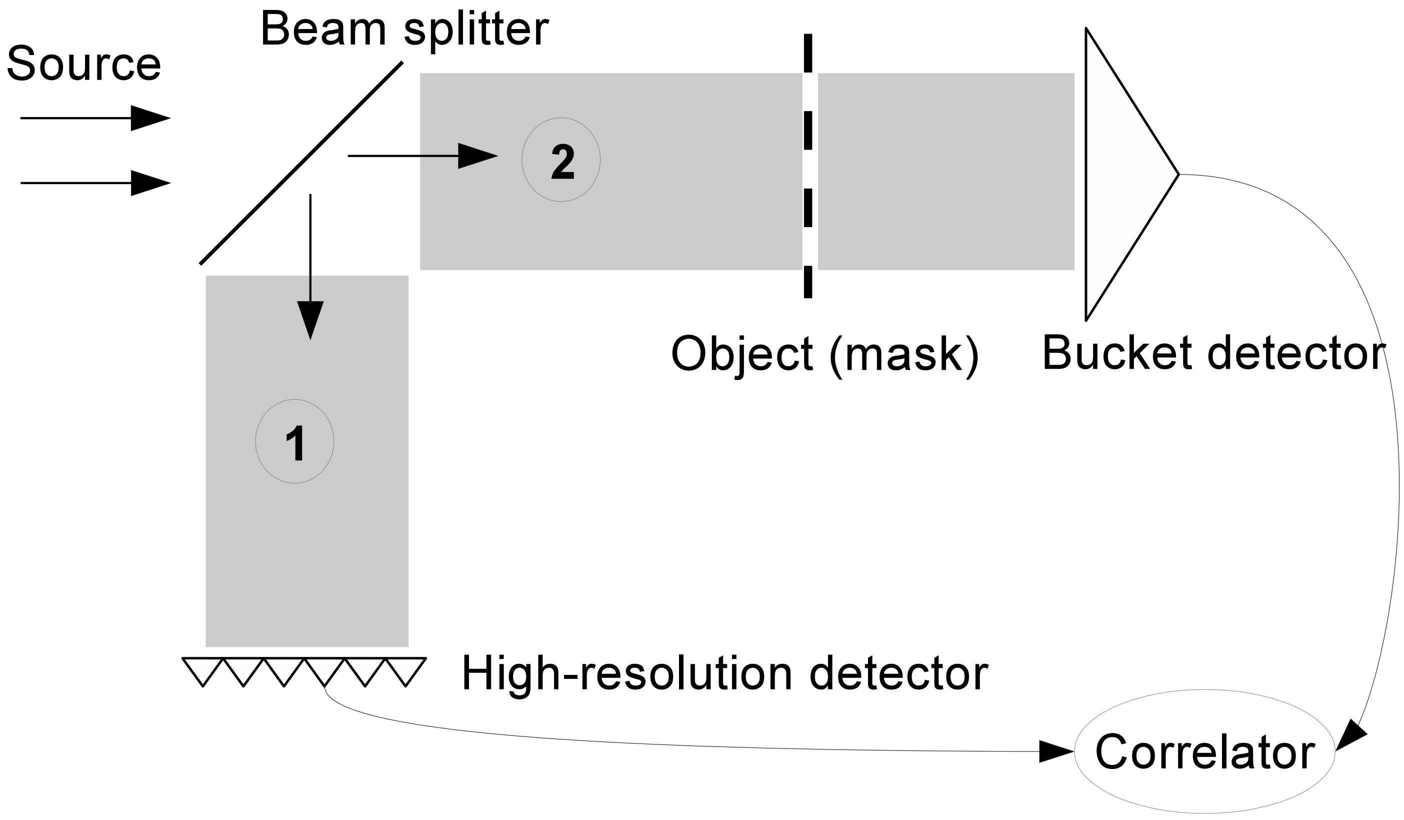
$$C^{(1)}(\boldsymbol{x}) = \int_{\mathbb{R}^2} \mathcal{H}(\boldsymbol{x}-\boldsymbol{y}) |\mathcal{T}(\boldsymbol{y})|^2 d\boldsymbol{y},$$

with

$$\mathcal{H}(\boldsymbol{x}) = \frac{r_o^4 \rho_{\rm gi0}^2}{2^8 \pi^2 L^4 \rho_{\rm gi2}^2} \exp\Big( - \frac{|\boldsymbol{x}|^2}{4\rho_{\rm gi2}^2}\Big), \qquad \rho_{\rm gi2}^2 = \rho_{\rm gi0}^2 + \frac{4c_o^2 L^3}{3\omega_o^2 Z_{\rm sca} \ell_{\rm c}^2}, \qquad \rho_{\rm gi0}^2 = \frac{c_o^2 L^2}{2\omega_o^2 r_o^2}$$

↪ Scattering only slightly reduces the resolution !
This imaging method is robust with respect to medium noise. It gives an image even when $L/Z_{\rm sca} \gg 1$.

# Ghost imaging in heterogeneous identical media




The medium in paths 1 and 2 is heterogeneous.

They are the *same realization*.

# Ghost imaging in heterogeneous identical media

- Resolution analysis in randomly heterogeneous and identical media.
- If the propagation distance is larger than the scattering mean free path, then

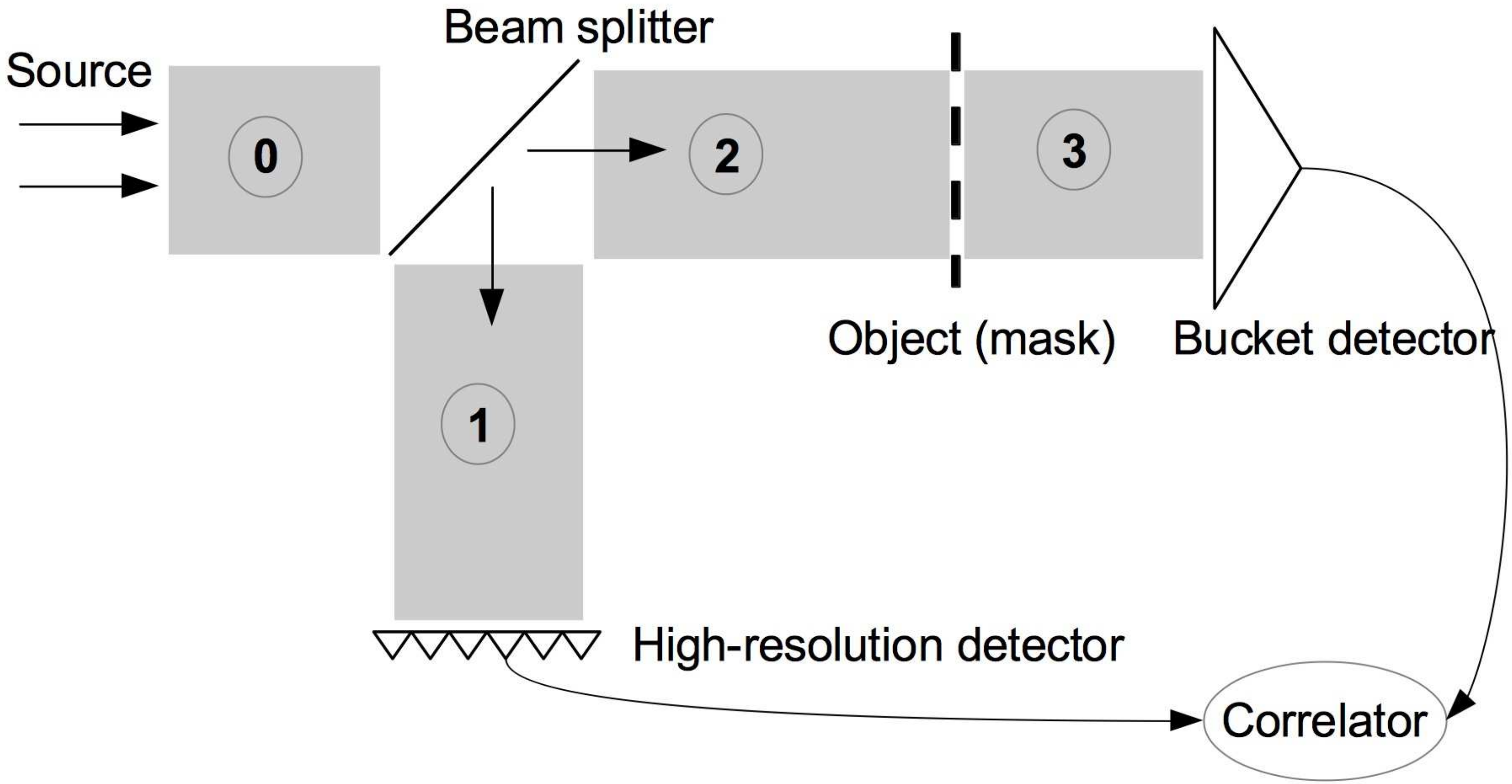
$$C^{(1)}(\boldsymbol{x}) = \int_{\mathbb{R}^2} \mathcal{H}(\boldsymbol{x}-\boldsymbol{y}) |\mathcal{T}(\boldsymbol{y})|^2 d\boldsymbol{y},$$

with

$$\mathcal{H}(\boldsymbol{x}) = \frac{r_o^4}{2^8 \pi^2 L^4} \exp\Big( - \frac{|\boldsymbol{x}|^2}{4\rho_{\rm gi3}^2} \Big), \qquad \frac{1}{\rho_{\rm gi3}^2} = \frac{1}{\rho_{\rm gi0}^2} + \frac{16L}{Z_{\rm sca}\ell_{\rm c}^2}$$

$\hookrightarrow$ the radius of the convolution kernel is reduced by scattering and can even be smaller than the Rayleigh resolution formula: enhanced resolution compared to the homogeneous case (similar phenomenon observed in time-reversal experiments) !

# On the role of the random medium




Random medium in region 0 is *good*.

Random medium in regions 1 and 2 is *bad* (unless they are the same realization).

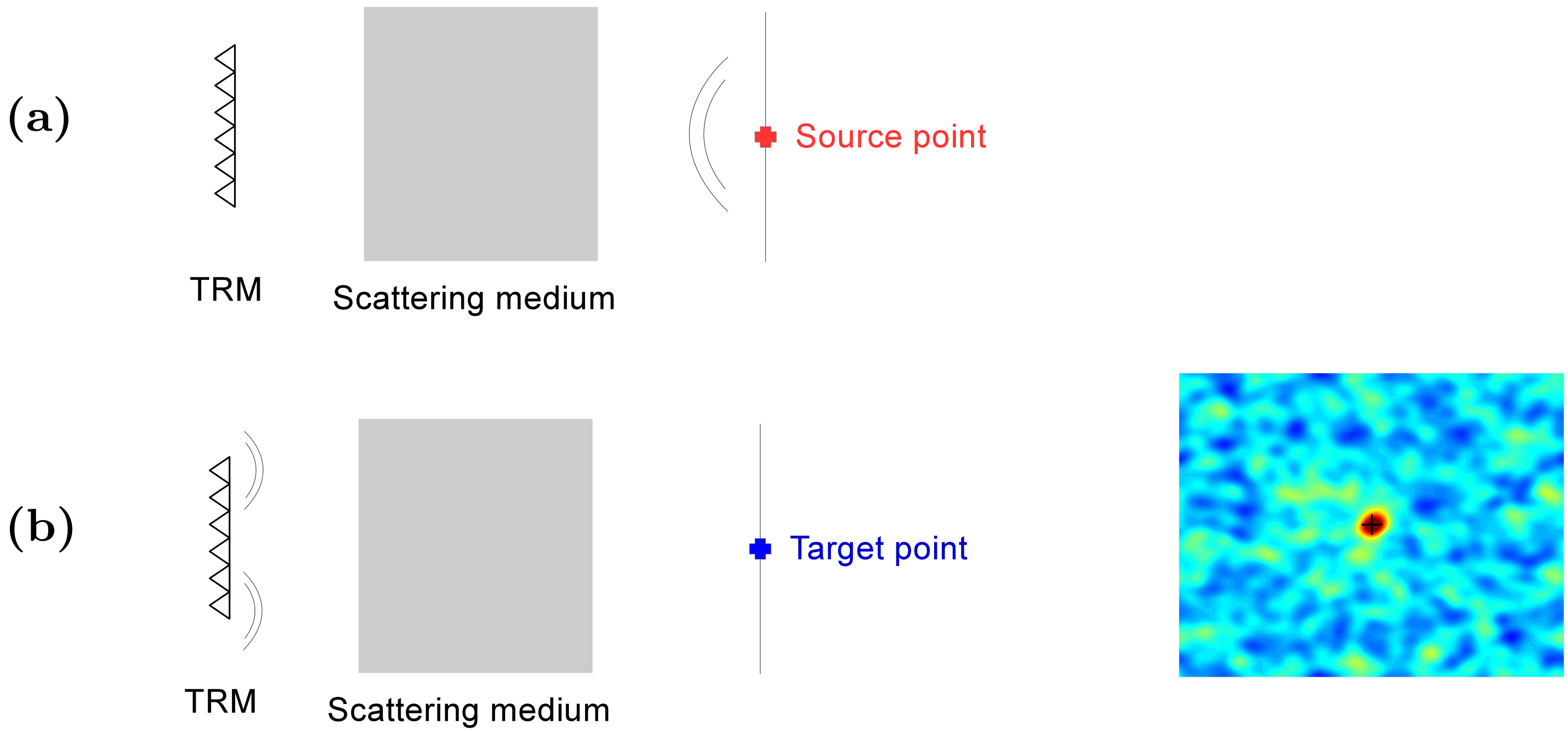
Random medium in region 3 plays *no role*.

## Optimal focusing

• Is there an optimal way of encoding a signal to counteract the corruption by the medium clutter ?

• Ideal case: send a probing signal from the target, record this, time reverse it and use it as a source.

# Optimal focusing in the "ideal" case with time reversal

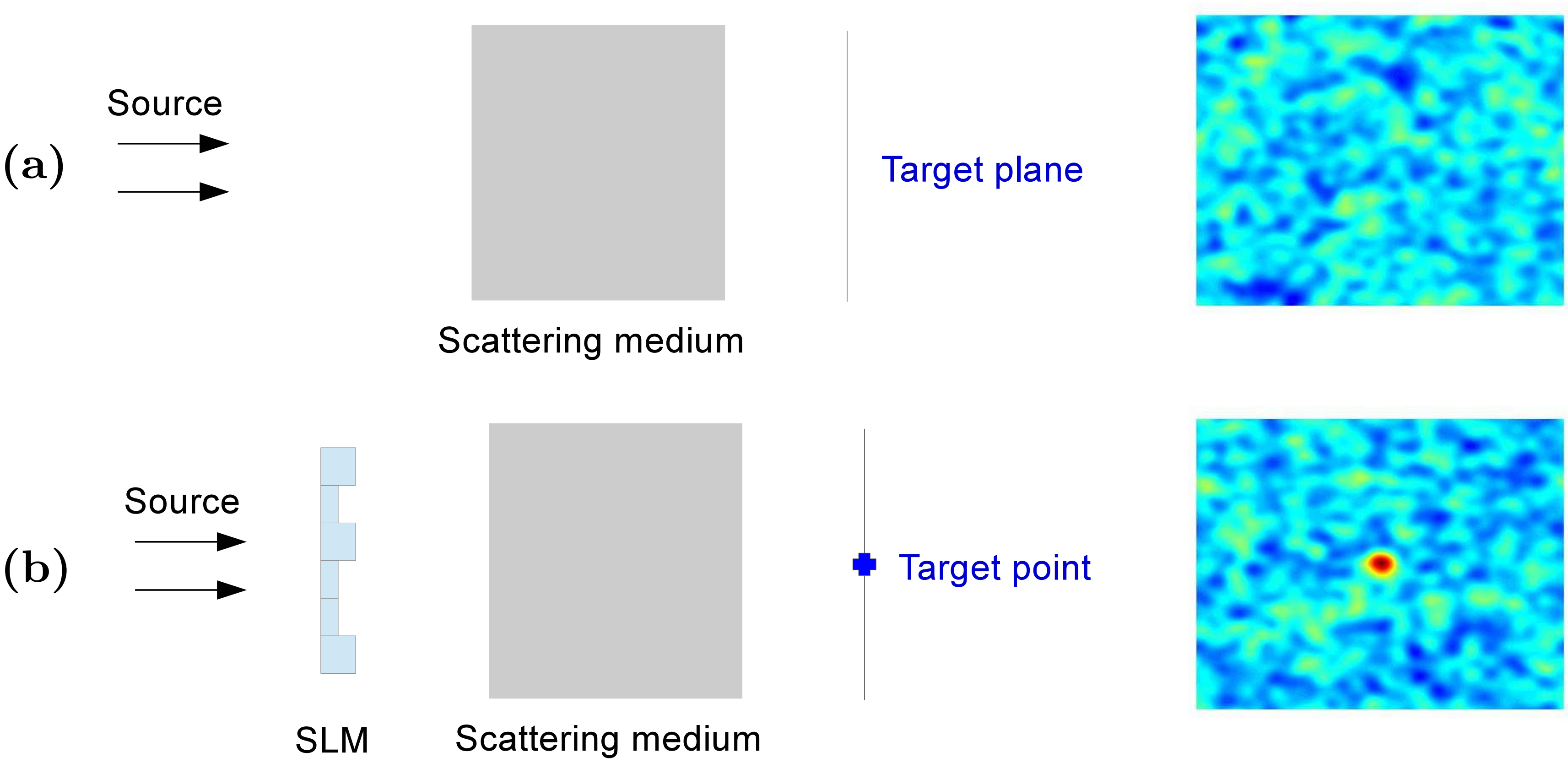



Time-reversal experiment through a scattering medium

(a) a point source emits a wave that propagates through the random medium and is recorded by the time-reversal mirror (TRM) used as receivers.

(b) the time-reversal mirror is used as an array of sources, it emits the time reversed (complex-conjugated in time harmonic case) recorded field, and the wave refocuses at the original source location.

# Optimal focusing with Spatial Light Modulator




Focusing wave through a scattering medium

Source: time-harmonic plane wave.

(a) Without any control one gets a speckle pattern in the target plane.

(b) With a spatial light modulator (SLM) one can focus on a target point by optimizing the phases of the elements [1].

[1] I.M. Vellekoop and A.P. Mosk, Opt. Lett. **32** (2007), 2309.

# Optimal focusing: Deep probing and focusing resolution

• When $L \gg Z_{\rm sca} = \frac{8c_o^2}{\gamma(\mathbf{0})\omega_o^2}$, the characteristic size or resolution $R_{\rm tr}$ of the refocused wave is

$$R_{\rm tr} \sim \frac{\lambda_o L}{\mathcal{A}_L} \sqrt{\frac{1}{6\pi^2} \frac{1+\frac{\mathcal{A}_L^2}{R_o^2}}{1+\frac{\mathcal{A}_L^2}{4R_o^2}}}$$

→ Effective time reversal aperture: $\mathcal{A}_L = \sqrt{\gamma(\mathbf{0})L^3/(6\ell_{\rm c}^2)}$ when $\gamma(\boldsymbol{x}) = \gamma(\mathbf{0})(1 - |\boldsymbol{x}|^2/\ell_{\rm c}^2 + ...)$.

• The focusing resolution corresponds to that of the Rayleigh resolution associated with the effective time reversal aperture.

• In a strongly scattering medium:

→ Focusing resolution depends only mildly on $R_o$, the radius of the SLM.

→ No dependence on $\rho_o$, the radius of the SLM elements (provided $\rho_o \ll \mathcal{A}_L$) !

→ However, SNR sensitive to $\rho_o$ !

## Optimal focusing: Signal-to-noise ratio with deep probing

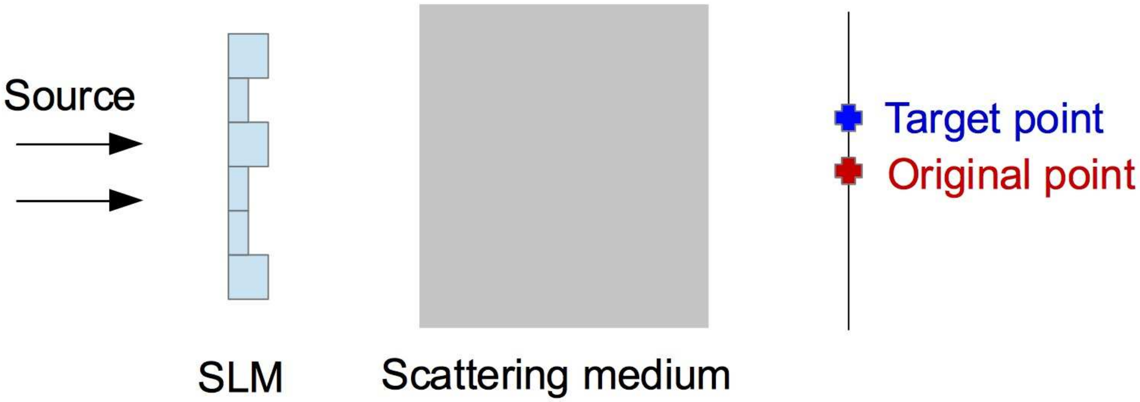
• Define signal-to-noise-ratio by $\mathrm{SNR} \equiv \mathbb{E}^2[\hat{u}]/\mathrm{Var}(\hat{u})$.

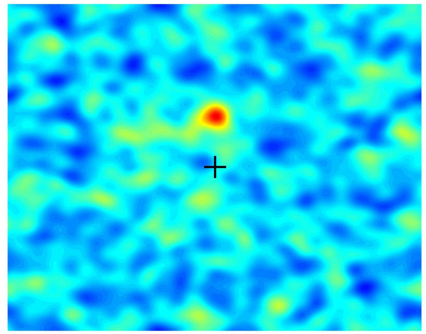
When $L \gg Z_{\mathrm{sca}}$,

$$\mathrm{SNR} = \frac{1 + (\mathcal{A}_L/\rho_o)^2}{1 + (\mathcal{A}_L/R_o)^2} \overset{\mathcal{A}_L \gg \rho_o}{\simeq} \left(\frac{\min(R_o, \mathcal{A}_L)}{\rho_o}\right)^2 = \begin{cases} \dfrac{\mathcal{A}_L^2}{\rho_o^2} & \text{if } \rho_o \ll \mathcal{A}_L \ll R_o, \\ \dfrac{R_o^2}{\rho_o^2} & \text{if } R_o \ll \mathcal{A}_L. \end{cases}$$

$\hookrightarrow$ SNR is the number of mirror elements $N$ needed to cover the effective mirror size $\min(R_o, \mathcal{A}_L)^2$.

# Optimal focusing: Steering a beam through clutter



Focusing wave through a scattering medium.

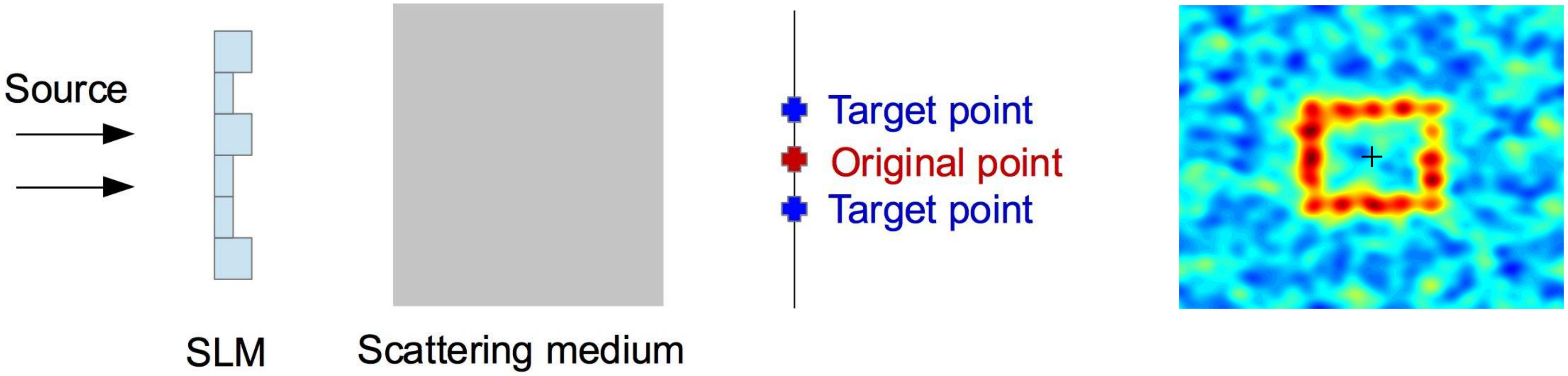
Focusing on a prescribed point in the neighborhood of the original target point. Impose an additional linear phase (the cross in the right image stands for the original target point).

→ Resolution as before, however, reduced signal-to-noise ratio due to de-correlation of wave paths (limited "memory" effect).

- Focusing region radius $R_{\rm max}$ is limited by SNR:

$$R_{\rm max}^2 \sim 3R_{\rm tr}^2 \frac{1}{1+(\mathcal{A}_L/R_o)^2} \ln \frac{1+(\mathcal{A}_L/\rho_o)^2}{1+(\mathcal{A}_L/R_o)^2}.$$

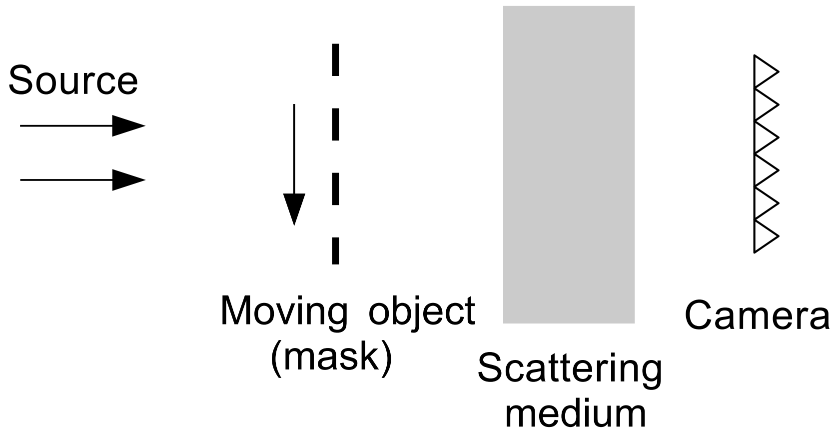
## Optimal focusing: Image transmission through clutter




### Transmission of an image

Here a square modeled as a set of sixteen target points is transmitted with the SLM based on one original target point. The cross in the right image stands for the original target point.

## Speckle intensity correlation imaging through a scattering medium




Experimental set-up [1]

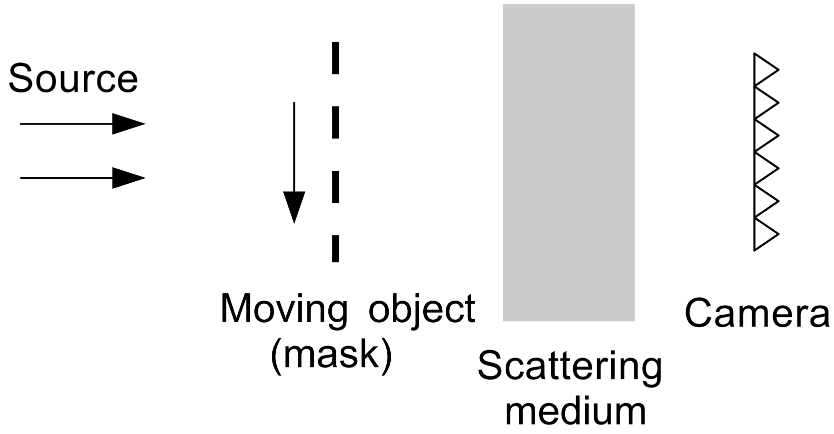
The source is a time-harmonic plane wave.

The object to be imaged is a mask that can be shifted transversally.

For each position of the object the spatial intensity of the transmitted field can be recorded by the camera.

[1] J. A. Newmann and K. J. Webb, PRL **113**, 263903 (2014).

## Speckle intensity correlation imaging through a scattering medium




- The field just after the object is of the form

$$U_{\boldsymbol{r}}(\boldsymbol{x}) = U(\boldsymbol{x} - \boldsymbol{r}),$$

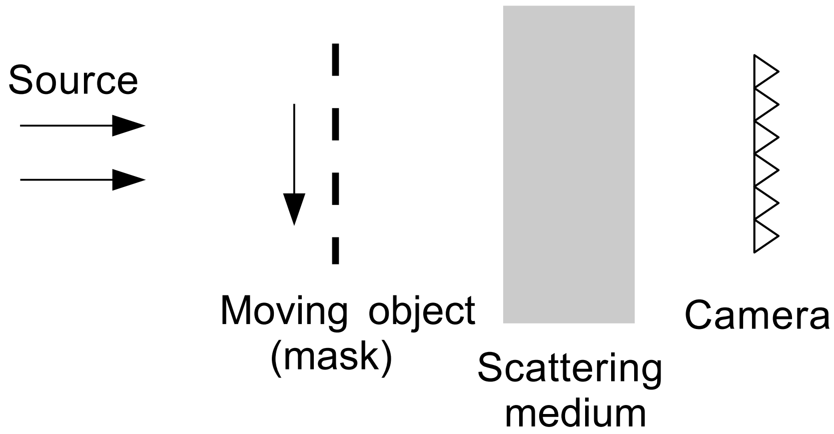
for some function $U$. The field in the plane of the camera is denoted by $E_{\boldsymbol{r}}(\boldsymbol{x})$.

- The measured intensity correlation is

$$\begin{aligned} C_{\boldsymbol{r},\boldsymbol{r}'} &= \frac{1}{|A_0|} \int_{A_0} |E_{\boldsymbol{r}}(\boldsymbol{x})|^2 |E_{\boldsymbol{r}'}(\boldsymbol{x})|^2 d\boldsymbol{x} \\ &\quad - \Big( \frac{1}{|A_0|} \int_{A_0} |E_{\boldsymbol{r}}(\boldsymbol{x})|^2 d\boldsymbol{x} \Big) \Big( \frac{1}{|A_0|} \int_{A_0} |E_{\boldsymbol{r}'}(\boldsymbol{x})|^2 d\boldsymbol{x} \Big), \end{aligned}$$

where $A_0$ is the spatial support of the camera.

# Speckle intensity correlation imaging through a scattering medium




- Result: When $L \gg Z_{\rm sca}$ and $\mathcal{A}_L \gg {\rm diam(camera)}$,

$$C_{\boldsymbol{r},\boldsymbol{r}'} \approx \Big| \int |\hat{U}(\boldsymbol{\kappa})|^2 \exp\big(i\boldsymbol{\kappa}\cdot(\boldsymbol{r}'-\boldsymbol{r})\big) d\boldsymbol{\kappa}\Big|^2,$$

up to a multiplicative constant, where

$$\hat{U}(\boldsymbol{\kappa}) = \int U(\boldsymbol{x}) \exp\big(-i\boldsymbol{\kappa}\cdot\boldsymbol{x}\big) d\boldsymbol{x}.$$

$\hookrightarrow$ It is possible to reconstruct the incident field $U$ by a phase retrieval algorithm.

## Conclusion

- Fourth-order moment of the wave field is useful.
- First application: Scintillation index and stability of Wigner transform.
- Second application: Intensity correlation-based imaging, ghost imaging.
- Third application: Optimal focusing through scattering medium.
- Hopefully, many other applications !