Electromagnetic and Topographic Coupling, and LOD Variations

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Abstract

The role played by different core processes in the changes in the Earth's rotation is assessed and fully dynamical models of the torsional Alfvén waves inside the fluid core are reviewed. These waves, first studied by Braginsky (1970), consist of geostrophic circulation. They have decadal periods and yield time changes in core angular momentum. They arise from small departures from an hypothetical quasi–static state, where the total action of the Lorentz force on the geostrophic cylinders cancels out. They cause torques acting on the mantle. Simple models of the torsional waves that rely only on zonal averages of the magnetic field have incorporated electromagnetic coupling to the mantle. They, however, need some correction. In addition, only a kinematic approach of the topographic coupling, caused by non–axial symmetry of the fluid cavity, has been successfully attempted to date. Taking into account uncertainties in the height of the core–mantle topography and in the electrical conductivity of the deep mantle, it turns out that, in the present state of core modelling, the pressure, gravity and electromagnetic torques acting on the mantle may all produce decade changes in the length of the day with a magnitude comparable to the observations.

1 Introduction

Our understanding of the part played by the core in the changes in the Earth's rotation has been much improved during the recent years because of longer and more accurate geodetic series, refined modelling of the atmospheric and oceanic contributions to the Earth's angular momentum budget and progress in dynamo theory. We have benefited also from a wealth of detailed studies about core–mantle coupling at different timescales. The areas of Earth's rotation studies where processes taking place in the Earth's core or at the core–mantle boundary (CMB) are important are now well delimited. As far as the Earth's spin rate is concerned, only decade and perhaps part of the longer period variations are caused by such processes. Motions in the Earth's core are probably inoperative in the excitation of the polar motion but a resonance with the quasi–diurnal inertial core mode of nearly rigid rotation about an equatorial axis plays an important part in the response of the Earth (nutations of its rotation axis) to gravitational torques from the moon and the sun. This review deals mainly with the origin of the decade variations in the length of day (l.o.d.) and the nutation problem is mentioned more briefly. The first question is indeed much more intricate and touches the theories of the geomagnetic secular variation and the geodynamo. However, the two problems should not be lightly dissociated since modelling of forced nutations may eventually constrain parameters (electrical conductivity of the lower mantle, energy of the small scale magnetic field at the core surface, CMB topography) that are crucial in the modelling of the decade variations in the Earth's spin rate. I review now other geophysical informations on these parameters.

Models of CMB topography have been inferred from shear and compressional velocities within the mantle: seismic anomalies are converted into density anomalies, which drive mantle convection and cause a dynamical topography of both the Earth's surface and the core–mantle boundary. The calculated height above a reference ellipsoid of the latter surface is of the order of a few km (Forte and Peltier, 1991; Defraigne et al., 1996). Equipotential surfaces of the gravity field are also obtained. At the Earth's surface, the observed geoid constrains the modelling, whereas Forte and Peltier (1991) (and respectively Defraigne et al. (1996)) inferred that the deviation from an ellipsoid of the gravity equipotential surface is of the order of 500 (150) m. at the CMB and of the order of 100 (30) m. at the inner core boundary. The electrical conductivity of most of the lower mantle is rather well known. Studies of electrical currents induced in the mantle by external fluctuations (magnetotelluric and magnetic observatory data) show that the conductivity is of the order of 1 Sm^{-1} at the top of the lower mantle (Schultz et al., 1993; Petersons and Constable, 1996) while high pressure experiments show that it is of the order of 1-10 Sm^{-1} in most of the lower mantle. The remaining incertainties stem from the dependence of the electrical conductivity on the temperature and the aluminium content (Xu et al., 1998). The combination of experiments with geophysical studies may yet narrow the range of permissible values (Dobson and Brodholdt, 2000). We shall see below (section 5) that the electrical currents circulating in the lower mantle with conductivity 5 $Sm^{-1}$are too weak to couple efficiently core and mantle on the decade timescale. Only the electrical currents at the bottom of the mantle may be intense enough to participate to core-mantle coupling, assuming a high conducting layer to be present there.

A variety of seismological investigations have been recently focused on the mantle region just above the CMB (see the collected articles in Gurnis et al. (1998)). The D" layer encompassing 200-300 km of the lowermost mantle has long ago been identified as a region of low seismic velocity gradients. It shows intense lateral variations, in particular of thickness, and it includes regions where seismic waves are anisotropic. In places, it is separated from the normal mantle by a velocity discontinuity (or an high velocity layer). An ultralow velocity zone (ULVZ) has recently been discovered (see the review of Garnero (2000)) at the bottom of D". Its thickness is 5-50 km where it has been detected (a third of the probed areas). The low velocity in this region may be the result of partial melting. This patchy zone may be much more dense than the surrounding mantle causing CMB topography. Only a weak perturbation of the gravity potential would be associated with such a topography. Increase of the electrical conductivity in the ULVZ is likely if it is partially molten. Its magnitude would depend on the chemical composition of the zone. These seismic observations may be taken as an indication of lateral variation of the electrical conductivity at the bottom of the mantle. Anisotropy of the conductivity is also a possibility.

Recently, seismologists have thoroughly investigated a possible differential rotation between the inner core and the solid mantle. Its determination would obviously yield an invaluable constraint on models of the Earth's rotation. Differential travel times of seismic waves and free oscillations have been examined, whereas Vidale et al. (2000) have used temporal changes in the scattering of seismic waves inside the inner core to suggest a differential rotation between the inner core and the mantle at 0.15 deg per year between 1971 and 1974 (see also the references inside this article). This method has the potential to monitor the inner core axial rotation for periods of a few years. Meanwhile, we do not know whether the detected rotation is steady or participates in the decadal changes in the Earth's axial rotation reflected in l.o.d. fluctuations.

Magnetic field observations remain the principal source of information on core dynamics. In this paper, I keep the usual terminology and I refer to the time series of the Earth's magnetic field as secular variation (s.v.) data. We are now expecting a dramatic improvement in the accuracy of s.v. data after a few years of continuous satellite recordings of the three components of the magnetic field, but, despite all the efforts of data analysis, we still have to rely heavily on dynamo models to unravel the mysteries of core dynamics. Rapid changes of the geostrophic velocity appear intertwined with a slow evolution of an important force equilibrium inside the core (Taylor, 1963). They entail changes in core angular momentum, which attest, conversely, the inner working of the Earth's dynamo. The timescales of the different mechanisms, inside the core, are measured against the magnetic diffusive time τ_d, of the order of a few tens of thousands of years. In the presence of rapid rotation and of a strong ambiant magnetic field, the long lengthscale waves riding inside the core (MC waves; Fearn et al. (1988)) have periods τ_{MC} of the order of τ_d/Λ, where the Elsasser number Λ gives the strength ratio of the magnetic force to the rotation force, on timescales comparable to τ_d and longer. Because Λ is probably of order unity, the general opinion is that these MC waves do not represent the observed rapid variations of the Earth's magnetic field. Thus, the Alfvén torsional waves, which involve only the geostrophic part of the motion, stand out because of their rapid periods (tens of years). They were first described 30 years ago by Braginsky (1970) in a spherical cavity. This review discusses how this theory has grown with the contact of the geophysical data that have since been collected. I rely also on recent investigations of the convective dynamo (Bell and Soward, 1996; Bassom and Soward, 1996) to suggest a possible extension to the case of bumpy core–mantle boundary and/or inner core surface. Finally, I remark that Alfvén torsional waves are not easily excited within the parameter range where fully consistent numerical models of the geodynamo currently operate. In these models, the ratio between the magnetic diffusivity η and the kinematic viscosity ν is decreased to enable dynamo action. As a result, the spin-up timescale τ_E becomes shorter than the period T_{TA} (see equation 29) of the torsional waves. Introducing the magnetic Prandtl number $P_m = \nu/\eta$, we obtain $T_A/\tau_E \simeq P_m^{1/2}\Lambda^{-1/2}$. With $P_m \geq 1$ and $\Lambda = O(1)$, viscous dissipation precludes propagation of the torsional waves. This situation contrasts with the actual geophysical case, where $P_m = O(10^{-6})$.

In the following section, I discuss how the core responds to torques of external origin. Next, I present the theory of torsional Alfvén waves, which gives an appealing explanation for the decade changes in the length of the day. A fourth and short section is devoted to extensions of this theory in presence of an inner core and of regions where geostrophic contours do not exist. However, simplicity goes only so far. From models of the secular variation of the Earth's magnetic field, it transpires that other core surface motions, besides Alfvén torsional waves, have also short timescales. We have to rely on kinematic theories, reviewed in §5, to evaluate their possible influence on core–mantle coupling.

2 Response of the core to changes in the rotation of its container

I begin with a summary of the changes in the Earth's rate of rotation and orientation in order to get the role of the core into perspective. It can be kept brief because very useful review articles, giving a lot of references, are already available (Eubanks, 1993; Dickey and Hide, 1991). There is strong evidence that exchanges of angular momentum between core and mantle occur on the decade time-scale (Jault et al., 1988; Jackson et al., 1993; Jackson, 1997). These yield variations in the length of the day of up to a few milliseconds (ms). On the other hand, the role of the atmosphere in rapid changes in the rotation rate of the solid Earth is well ascertained (Rosen et al. (1990) and references therein) up to periods of a few years. The oceans and the atmosphere may play also some role on longer periods. As an example, Abarca del Rio (1999) argued that thermal expansion of the oceans may have caused an increase in the l.o.d. as large as 0.25 ms from 1950. Yet, the contributions of the outer fluid envelopes to the Earth's angular momentum budget, on the decade timescale, are minor. It is likely that the core does not play much role at shorter periods but we lack quantitative studies. In order to shed light on this question, it is possible either to look for discrepancies between changes in the combined angular momentum of the atmosphere and the oceans and changes in the angular momentum of the solid Earth on periods of a few years (see e. g. Abarca del Rio et al. (2000)) or to study the response of the core to forcing by changes in the rotation rate of the mantle (spin-up, spin-down).

Zonal tides produce changes in the Earth's axial moment of inertia and in the angular momentum of oceanic currents. Variations in the spin-rate of the mantle ensue. From an analysis of the tidally induced changes in the l.o.d., Dickman and Nam (1998) concluded that the core is fully decoupled from the mantle at 9 days period, while they found that no firm conclusions are possible from the study of longer period tidal effects in the Earth's rotation. Up to the annual period, once the atmospheric contribution is removed and assuming core–mantle decoupling, the residual l.o.d. is only a few percents of the total l.o.d. signal. Thus, full coupling of the core with the mantle at these frequencies would be measurable since the moment of inertia of the core is 12% of that of the mantle. Then, Zatman and Bloxham (1997a) remarked that the characteristic timescale of the coupling processes between the core and the mantle can be inferred, if it is short enough, from a study of the phase difference between l.o.d. and atmospheric angular momentum. Such a phase difference has long been sought as a clue of oceanic influences. The first studies (Eubanks et al., 1985; Rosen et al., 1990) found no difference between the two series. With more accurate geodetic data, the coherence between the two series has been increased to the level where oceanic contributions are now significant. Taking advantage of the improved modelling of oceanic dynamics, Dickey et al. (2000) have been able to combine the angular momentum of the atmosphere and of the oceans and to compare the resulting series with l.o.d. They have found no phase difference. It would imply that, up to the annual period at least, changes in the rate of rotation of the mantle have no significant effects on the core rotation. Finally, Dickman (2001) has just advocated incorporating, in models of l.o.d. changes at annual and semi-annual periods, the variations in the Earth's gravity inferred from satellite laser ranging observations. He thus expects to measure accurately the extent of core–mantle coupling at annual period. In the meantime, I tentatively conclude from this discussion that the characteristic times of the coupling mechanisms between core and mantle are of the order of one year at least.

Accurate VLBI measurements of the nutations of the Earth's axis of rotation may give also constraints on core–mantle coupling. The VLBI series is now long enough to determine the parameters of the 18.6–year nutation. If the core were inviscid, its outer boundary spherical and the lower mantle electrically insulating, forced nutations of the Earth would yield a differential rotation between the core and its container about an equatorial axis fixed in an inertial frame. This diurnal mode, in a frame attached to the mantle, is actually the simplest possible free mode of a spherical fluid body: the "tilt-over" mode. Since the core–mantle boundary is ellipsoidal, this free mode is coupled to the mantle, it is a normal mode of the whole Earth and it entails a rotational motion of the solid Earth, the free core nutation, also observed in VLBI series. The period of this free mode (retrograde in the mantle frame) depends on the core ellipticity. In the mantle frame, the mode induces resonance in the diurnal tides. In an inertial frame, its period is close to one year and the forced retrograde annual nutation of the Earth is significantly modified because of the presence of the fluid core. Thus, the measurement of the amplitude of this forced nutation, together with analysis of tidal gravity data, enables to determine accurately the period of this free core mode of rotation about an equatorial axis, which, in turn, constrains the oblateness of the core-mantle boundary (Gwinn et al., 1986; Neuberg et al., 1987). In the same way, another free mode (prograde) consists mainly of the rotation of the solid inner core about an equatorial axis. It has a longer period in an inertial frame and influences the 18.6–years nutation. Finally, the motive for this discussion is that observations of forced nutations out-of-phase with luni-solar forcing require, according to Buffett (1992), very efficient electromagnetic coupling at the diurnal frequency. In particular, the out-of-phase component of the annual retrograde nutation is very sensitive to core–mantle coupling and cannot be explained by mantle anelasticity or ocean tide loading (Mathews et al., 2001). It constrains the conductivity of a thin layer above the core–mantle boundary of thickness the penetration depth of diurnal signals from the core. Buffett et al. (2000) have just reported the result of an inversion of the most up-to-date nutation observations. The model includes a solid layer of core conductivity attached to the mantle and an energetic small scale magnetic field, such that the radial field has an uniform r.m.s. strength of $7.1 \times 10^{-4} T$. over the CMB. The remaining residuals after the inversion are small but some of them may still be significant (the out-of-phase component of the prograde 18.6 years nutation). The hypothesis of high energy in the small scale part of the magnetic field will also soon be assessed with satellite observations of the geomagnetic field (Neubert et al., 2001). Finally, the model is compatible with the long spin-up time for the core advocated in the previous paragraph and with short diffusion times of electromagnetic signals through the mantle, yet the two hypotheses of highly conducting solid layer and strong small scale magnetic field at the core surface are far from trivial. We must allow for other possibilities. In addition to the nearly rigid rotation about an equatorial axis, other inertial modes of the fluid outer core are coupled to the nutations of the solid Earth when there is topography at the CMB (Wu and Wahr, 1997). The modes with nearly diurnal periods are particularly significant. The difficulty, here, is to avoid introducing too many parameters for too few data. I conclude now this summary of the constraints that the externally driven changes in the Earth's rotation give on core–mantle coupling with a discussion of another free mode of the Earth, the Chandler wobble.

First, there is a sharp contrast with the free core nutation problem. Dissipative processes at the core–mantle boundary, investigated as a damping mechanism of the Chandler wobble, can be neglected compared to mantle anelasticity (Smith and Dahlen, 1981). On the other hand, a possible role of the core in the excitation of this wobble of the Earth's rotation axis has long been debated. The Chandler excitation power shows indeed dramatic decadal changes, which have been tentatively associated with core processes such as impulses in the secular variation of the Earth's magnetic field (Gibert et al., 1998). There is however growing evidence from improved modelling of oceanic circulation that the Chandler wobble is mostly excited by a combination of atmospheric and oceanic processes. Celaya et al. (1999) relied on a statistical analysis of a full climate model coupling the oceans and atmosphere to infer that the climate excitation of the Chandler wobble has the right amplitude and timescale. They noticed indeed that the excitation of the wobble is consistent with a stationary

Gaussian process. In a complementary study, Gross (2000) used a global oceanic circulation model, constrained with actual observations, to argue that fluctuations in the pressure at the bottom of the oceans, driven by surface winds, have been the main excitation process during 1985-1996. He assumed the same value of the quality factor (Q=179) of the Chandler wobble as Celaya et al. (1999), which is still uncertain. On longer timescales, the influence of core dynamics is plausible. However, estimates of the pressure torque acting from the core seem too small (Hulot et al., 1996; Hide et al., 1996) to explain past motions, with decadal periods, of the pole. In addition, the most recent and accurate data do not show decade variations (Mc Carthy and Luzum, 1996).

3 Modelling the Alfvén torsional waves inside the Earth's core

Alfvén torsional waves consist of geostrophic motions, which carry axial angular momentum. They have periods adequate to participate in the l.o.d. variations with decade timescales and they occur naturally as the configuration of the magnetic field slowly evolves. Keeping only the necessary ingredients of a convective dynamo model (see e.g. equation (3.36) of Fearn (1998)), the momentum equations for slow and large scale motions are:

$$2\rho\,(\Omega \times \mathbf{u}) = -\nabla p + \mathbf{j} \times \mathbf{B} - \alpha\rho\Theta\mathbf{g} \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{2}$$

$$\mathbf{u} \cdot \mathbf{n} \,|_{\Sigma} = 0 \tag{3}$$

where $\mathbf{g}$ is the gravity acceleration, Θ is the temperature, α is the coefficient of thermal expansion, $\mathbf{B}$ is the magnetic field, $\mathbf{j}$ is the electrical current density, p is the pressure, $\mathbf{u}$ is the velocity, Ω is the spin rate of the mantle, ρ is core density, and $\mathbf{n}$ is the outward normal to the boundary Σ of the fluid volume. The first equation represents the magnetostrophic balance between the rotation and Lorentz forces. Many terms are neglected and we shall see that it is not always consistent. Knowing $\mathbf{B}$ and Θ , $\mathbf{u}$ is determined up to an arbitrary geostrophic motion $\mathbf{u}_g$, obeying the balance:

$$2\rho\,(\Omega \times \mathbf{u}_g) = -\nabla p_g, \quad \mathbf{u}_g \cdot \mathbf{n} \,|_{\Sigma} = 0 \tag{4}$$

Geostrophic motions are independent of the coordinate z in the direction of the rotation axis. They are thus entirely defined by their streamlines on Σ, the pair of geostrophic contours Γ. Denote respectively z_T and z_B the z-coordinates along each upper and lower geostrophic contours. The length $H = z_T - z_B$ is an invariant of each pair of contours. The pressure p_g is constant on each cylinder $\mathcal{C}$, parallel to the rotation axis, generated by geostrophic contours. These cylinders are defined in an unique way by their total height H. There may be regions of the core where no geostrophic contours exist. I defer their discussion to section (4). The Taylor's constraint on the right-hand side of (1) is a corollary of the non-uniqueness of $\mathbf{u}$:

$$\forall \mathbf{u}_g, \quad \int_{V_\Gamma} \mathbf{u}_g.\,(\mathbf{j}\times\mathbf{B} - \alpha\Theta\mathbf{g})\, dV = 0\,, \tag{5}$$

where V_Γ denotes the region where geostrophic contours exist. By investigating the volume comprised between the cylinders $\mathcal{C}(H)$ and $\mathcal{C}(H+dH)$, it can be checked that the generalized version of the Taylor's condition given by Bassom and Soward (1996) (their equation (1.3)) is equivalent to (5). In the special case of a spherical cavity enclosed between two spheres of radius respectively b and a, the geostrophic contours are circular, the geostrophic velocity is constant on each cylinder $\mathcal{C}$, and

$$z_B = -z_T, \quad \mathbf{u}_g = u_g(s)\mathbf{e}_\phi \qquad \text{at} \quad b \le s \tag{6}$$

$$\mathbf{u}_g = u_g^{\pm}(s)\mathbf{e}_\phi \qquad \text{at} \quad s\le b \quad \text{and} \quad \pm\, z \ge 0 \tag{7}$$

(where s is the distance to the rotation axis and $\mathbf{e}_\phi$ is the unit azimuthal vector). Let us scale the different surface contributions to (5) with respect to the JB Taylor volume integral derived for an insulating mantle and a spherical core, radius a,

$$\forall \mathbf{u}_g, \quad \int \mathbf{u}_g.\,(\mathbf{j}\times\mathbf{B})\, dV = 0\,, \tag{8}$$

with $\mathbf{j} = 0$ in the solid mantle: I estimate the relative error caused, on the long geodynamo timescale, by the substitution of (8) to (5). Assuming that buoyancy and electromagnetic forces are similar in strength, topographical effects, which arise because geostrophic contours deviate from perfect circles, scale as:

$$\delta = \frac{h}{a} = 3\times 10^{-4}, \tag{9}$$

where the height h of the CMB topography yields also the characteristic distortion of the cylinders. The importance of the Lorentz force acting in the lower mantle compared to its counterpart inside the core is measured by:

$$\frac{\sigma_m \Delta}{\sigma_c a} = 3\times 10^{-5}, \tag{10}$$

where Δ is the thickness of the layer of conductivity σ_m at the bottom of the mantle and σ_c is core conductivity. The viscous drag, scaled by the square root of the Ekman number E is also negligible:

$$E^{1/2} = 10^{-7}, \quad E = \frac{\nu}{\Omega a^2}. \tag{11}$$

Core–mantle coupling thus appears unimportant for these long timescale dynamics.

When condition (5) is not fulfilled, the equation (1) has no solutions and it has to be modified by including at least another term, such as inertia or the viscous force (see the complete discussion by Roberts and Soward (1972)). I shall leave out a possible turbulent friction force at the solid boundaries (see the analysis of Desjardins et al. (2001) for an insulating mantle). The friction term is important in the atmosphere, where the wind speed can be measured at different heights above the lower surface, but its strength is difficult to estimate at the fluid core boundaries. In addition, its role may be taken over in the core case by the electromagnetic force. We shall see indeed that electromagnetic coupling with the mantle appears as a friction term in the equation of torsional waves (within our restrictive hypotheses on the distribution of mantle conductivity). Assuming that on short timescales inertia predominates over viscous friction, the equation for the rapidly varying part $\mathbf{v}_g$ of the geostrophic velocity $\mathbf{u}_g$ is:

$$\rho H \oint \frac{\partial \mathbf{v}_g}{\partial t} . d\Gamma = \int_{z_B}^{z_T} \left(\oint \left(\mathbf{j} \times \mathbf{B} - \alpha \Theta \mathbf{g} - \rho \frac{d\Omega}{dt} \times \mathbf{r} \right) . d\Gamma \right) dz \qquad (12)$$

(anticipating possible fluctuations of the spin rate of the mantle and noting the position vector $\mathbf{r}$). In turn, a magnetic field $\tilde{\mathbf{b}}$, with the same characteristic time as the geostrophic velocity, is induced:

$$\frac{\partial \tilde{\mathbf{b}}}{\partial t} = \nabla \times (\mathbf{v}_g \times \mathbf{B}) , \qquad (13)$$

in the interior of the core, where the time changes of $\tilde{\mathbf{b}}$ are fast enough to make diffusion negligible. The Lorentz force seeks to return each geostrophic cylinder to its stable state, defined by (5) and (1), and torsional Alfvén waves arise.

I find it useful to derive once again (see Braginsky (1970) and Roberts and Soward (1972)) the torsional waves equation in order to discuss recent studies and to plan future works. Furthermore, Fearn and Proctor (1992) remarked that manipulations of the Lorentz force integral over a geostrophic cylinder that are very useful in the axisymmetrical case are not easily generalized to non–axisymmetrical fields. Taking into account the non–axisymmetrical component of the magnetic field at the core surface adds indeed a minor complication to the equation, which I think is best to be made explicit.

Braginsky (1970) derived the equations for the torsional Alfvén waves in the spherical case (equation (6) with $b = 0$). The buoyancy contribution to the left-hand side of (12) vanishes:

$$4\pi \rho s^2 z_T \frac{\partial (\omega_g + \Omega)}{\partial t} = \int_{-z_T}^{z_T} \oint (\mathbf{j} \times \mathbf{B})_\phi \, s d\phi dz, \qquad (14)$$

where $\mathbf{v}_g = s\omega_g \mathbf{e}_\phi$, see (6). Braginsky assumed that a quasi–static state $(\mathbf{u}, \mathbf{B})$ exists and he considered $\tilde{\mathbf{b}}$ (see (13)) as a small perturbation. Indeed, the condition (8), which is fulfilled in the quasi–static basic state, makes possible to linearize (14). The equation (13) does not hold at the boundary. There, a magnetic diffusion layer is set up to match the magnetic field induced in the core interior to the magnetic field in the mantle. It is convenient to study separately the contributions of the interior field $\tilde{\mathbf{b}}$ and of the diffusion layer field $\tilde{\mathbf{b}}_\lambda$ to

(14). Equation (13) gives, in the interior of a spherical core:

$$\frac{\partial \tilde{\mathbf{b}}}{\partial t} = B_s s \frac{\partial \omega_g}{\partial s} \mathbf{e}_\phi - \omega_g \frac{\partial_1 \mathbf{B}}{\partial \phi} \tag{15}$$

where $\partial_1/\partial\phi$ denotes differentiation with respect to ϕ holding $\mathbf{e}_s$ and $\mathbf{e}_\phi$ fixed. In turn, equation (15) gives the radial magnetic field at the bottom of the mantle $\tilde{b}_{mr}$ since the radial field is continuous across the magnetic diffusion layer. In a first stage, I suppose that the mantle is electrically insulating. Then, knowing $\tilde{b}_{mr}$ everywhere on the core surface, we deduce the two other components of the magnetic field $\tilde{\mathbf{b}}_m$ at bottom of the mantle. Finally, we have:

$$\tilde{\mathbf{b}} + \tilde{\mathbf{b}}_\lambda = \tilde{\mathbf{b}}_m \text{ at } r = a \tag{16}$$

This condition determines $\tilde{\mathbf{b}}_\lambda$.

Let us first study the contribution of the interior magnetic field to the right-hand side of (14). It is useful to remark that

$$\oint (\mathbf{j} \times \mathbf{B})_\phi \, d\phi = \frac{1}{s\mu_0} \oint \nabla . \left(s \mathbf{B}_M B_\phi \right) d\phi \tag{17}$$

where μ_0 is magnetic permeability, and $\mathbf{B}_M$ is the meridional magnetic field:

$$\mathbf{B}_M = \mathbf{B} - B_\phi \mathbf{e}_\phi \tag{18}$$

Equation (17) gives:

$$\begin{aligned} \int_{-z_T}^{z_T} \oint (\mathbf{j} \times \mathbf{B})_\phi \, d\phi dz \quad &= \quad \frac{1}{s^2 \mu_0} \frac{\partial}{\partial s} \left(s^2 \int_{-z_T}^{z_T} \oint B_s B_\phi d\phi dz \right) \\ &+ \frac{a}{\mu_0 z_T} \left(\oint B_r B_\phi d\phi \left(s, z_T \right) + \oint B_r B_\phi d\phi \left(s, -z_T \right) \right) \end{aligned} \tag{19}$$

Using $\tilde{\mathbf{b}} \ll \mathbf{B}$, we separate a volume term

$$I = \frac{1}{s^2 \mu_0} \frac{\partial}{\partial s} \left(s^2 \int_{-z_T}^{z_T} \oint \left(B_s \tilde{b}_\phi + B_\phi \tilde{b}_s \right) d\phi dz \right) \tag{20}$$

and a surface term

$$J = \frac{a}{\mu_0 z_T} \oint \left(\left(B_r \tilde{b}_\phi + B_\phi \tilde{b}_r \right) (s, z_T) + \left(B_r \tilde{b}_\phi + B_\phi \tilde{b}_r \right) (s, -z_T) \right) d\phi \tag{21}$$

After taking the time-derivative of (20), we can elininate $\tilde{\mathbf{b}}$ through the use of (15). We obtain

$$\frac{\partial I}{\partial t} = \frac{4\pi}{s^2 \mu_0} \frac{\partial}{\partial s} \left(z_T s^3 \frac{\partial \omega_g}{\partial s} \left\{ B_s^2 \right\} \right) \tag{22}$$

where $\left\{ B_s^2 \right\}$ is a measure of the square of the s-component of the magnetic field averaged on each geostrophic cylinder:

$$\left\{ B_s^2 \right\} (s) = \frac{1}{4\pi z_T} \int_{-z_T}^{z_T} \oint B_s^2 d\phi dz \tag{23}$$

Anticipating the final result, I note J_λ the contribution of the magnetic field of the diffusion layer to the right-hand side of (14):

$$J_\lambda = \frac{1}{\mu_0}\int_{-z_T}^{z_T}\oint B_r \frac{\partial \tilde{b}_{\lambda\phi}}{\partial r} d\phi dz = \frac{1}{\mu_0}\int_{-z_T}^{z_T}\oint \frac{B_r}{\cos\theta}\frac{\partial \tilde{b}_{\lambda\phi}}{\partial z} d\phi dz \tag{24}$$

or

$$J_\lambda = \frac{a}{\mu_0 z_T}\oint \left(\left(B_r \tilde{b}_{\lambda\phi}\right)(s, z_T) + \left(B_r \tilde{b}_{\lambda\phi}\right)(s, -z_T)\right) d\phi \tag{25}$$

By equation (16), we finally obtain:

$$J + J_\lambda = \frac{a}{\mu_0 z_T}\oint \left(\left(B_r \tilde{b}_{m\phi} + B_\phi \tilde{b}_{mr}\right)(s, z_T) + \left(B_r \tilde{b}_{m\phi} + B_\phi \tilde{b}_{mr}\right)(s, -z_T)\right) d\phi \tag{26}$$

With this expression, the equation for torsional Alfvén waves in a full sphere enclosed in an electrically insulating mantle is now complete. I write it below (equation 28) allowing for a thin layer of conducting material at the bottom of the mantle.

Studies of the Alfvén torsional waves riding inside the core may eventually lead to an assessment of the strength of the different torques acting at the CMB (Buffett, 1998). In particular, the electromagnetic torque has been thoroughly investigated. It is an obvious candidate as the damping mechanism of the waves whilst the viscous torque is usually neglected on the basis of its long timescale. Zonal motions at the core surface shearing an axisymmetrical magnetic field $\mathbf{B}_M(r,\theta)$ induce meridional electrical currents $\mathbf{j}_M(r,\theta)$ at the bottom of the conducting mantle. The resulting Lorentz force $\mathbf{j}_M \times \mathbf{B}_M$ is directed along $\mathbf{e}_\phi$ and exerts an axial torque on the mantle (see also section 5 below) . Thus, it is possible that a model of the Alfvén torsional waves, even one including only the interaction with the axisymmetrical part of the quasi–static magnetic field, yields an efficient electromagnetic torque acting on the mantle. I suppose that there is a thin conducting layer of conductivity $\sigma_m \exp(-r/\Delta)$ at the bottom of the mantle. Following the considerations alluded to in the Introduction, I take the conductance of the layer $\sigma_m \Delta(\theta,\phi)$ as laterally varying. The azimuthal magnetic field at the core surface is now $\left(\tilde{b}_{m\phi} + \tilde{b}_{\Delta\phi}\right)$, where $\tilde{b}_{\Delta\phi}$ denotes the azimuthal field induced by the shear at the core surface, the notation $\tilde{\mathbf{b}}_m$ being saved for the magnetic field at bottom of the insulating volume inside the mantle. Assuming that $(\sigma_m\Delta \ll \sigma_c\delta_\lambda)$, where δ_λ is the thickness of the diffusion layer, and that $\Delta \ll \delta_H$, where δ_H is the length scale of variation of B_r at the core surface, we obtain by continuity of the electrical field parallel to the boundary:

$$-\frac{\tilde{b}_{\Delta\phi}\mid_{r=a}}{\mu_0\sigma_m\Delta} = s\omega_g B_r \ . \tag{27}$$

Finally, the equation for Alfvén torsional waves is:

$$\rho s z_T \frac{\partial^2(\omega_g + \Omega)}{\partial t^2} = \frac{1}{s^2\mu_0}\frac{\partial}{\partial s}\left(z_T s^3 \frac{\partial \omega_g}{\partial s}\left\{B_s^2\right\}\right) \tag{28}$$

$$-\frac{as}{4\pi z_T}\left(\oint \sigma_m \Delta B_r^2 d\phi\,(s,z_T)+\oint \sigma_m \Delta B_r^2 d\phi\,(s,-z_T)\right)\frac{\partial \omega_g}{\partial t}$$

$$+\frac{a}{4\pi\mu_0 z_T}\oint\left(\left(B_r\frac{\partial \tilde{b}_{m\phi}}{\partial t}+B_\phi\frac{\partial \tilde{b}_{mr}}{\partial t}\right)(s,z_T)+\left(B_r\frac{\partial \tilde{b}_{m\phi}}{\partial t}+B_\phi\frac{\partial \tilde{b}_{mr}}{\partial t}\right)(s,-z_T)\right)d\phi$$

In the axisymmetrical case, there is no azimuthal magnetic field at the core surface and the last term disappears. Braginsky (1970) suggested that this term can also be neglected on the ground that, at the core surface, in his geodynamo model (Braginsky, 1964), the non–axisymmetrical part of the magnetic field is small compared to the axisymmetrical part. In the general case, whilst an expression of $\partial\tilde{b}_{mr}\mid_{r=a}/\partial t$ as a function of ω_g is directly obtained from (15), the determination of $\partial\tilde{b}_{m\phi}\mid_{r=a}/\partial t$ necessitates an integration over the entire core surface. Finally, operating with $4\pi\int_0^a s^2 ds$ on (28) yields the time derivative of the torque budget, and as a consequence the equation that determines $d\Omega/dt$; the last term of (28) does not contribute. Of course, the torsional Alfvén waves have larger amplitude where $\left\{B_s^2\right\}(s)$ is weak. The Taylor's condition (5) is obeyed on timescales long compared to the period of the torsional waves, which is of the order of:

$$\tau_{TA}=\frac{(\mu_0\rho)^{1/2}a}{\{B_s^2\}^{1/2}}, \tag{29}$$

where the denominator loosely refers to a typical value of $\left\{B_s^2\right\}(s)$. Braginsky adjusted this parameter to recover the characteristic timescale of the l.o.d. variations, which is 60 years in his opinion. The frequency $\varpi=2\pi\tau_{TA}^{-1}$ of the waves does not enter (28). Its introduction represents an important simplification in the writing of the equations only when the model includes a solid and conducting inner core. Then, we need to know the penetration depth in the inner core to avoid solving the induction equation there. Assuming that the timescale of the slow and large scale motions governed by equation (1) is of the order of the period $\tau_{MC}=\tau_d/\Lambda$ of the MC–waves (see the Introduction), we can check the consistency of the approach:

$$\frac{\tau_{TA}}{\tau_{MC}}=\left(\frac{B}{B_s}\right)\frac{1}{\tau_A\Omega}$$

where τ_A is the period of the Alfvén waves that would exist inside the core in the absence of rotation (given by the expression (29) with B substituted for B_s). Here, in a diffusionless situation, magnetic and rotation effects are compared by the very small parameter $(\tau_A\Omega)^{-1}$. This result validates the assumption that the waves that arise when condition (5) is not satisfied consist of geostrophic motions.

Modelling of the torsional Alfvén waves has also been encouraged by the successful interpretation of core angular momentum changes as the result of acceleration of the geostrophic motions (Jault et al., 1988; Jackson et al., 1993; Hide et al., 2000; Pais and Hulot, 2000). Models of zonal core surface velocities symmetrical with respect to the equatorial plane $u_\phi(\theta,t)$ ($\theta\leq\pi/2$ is colatitude) have been extracted from models of time–dependent core surface motions obtained after the inversion of secular variation data (in the mantle reference frame). In turn, $u_\phi(\theta, t)$ has been assimilated to $v_g(s)$. An estimate of time changes of core angular momentum $\mathcal{A}_c$ follows

$$\frac{d\mathcal{A}_c}{dt} = 4\pi\rho\frac{d}{dt}\int s^3 z_T\left(\omega_g(s) + \Omega\right) ds \tag{30}$$

and has been used to check, with l.o.d. data, that core and mantle form a closed system on the decadal timescale:

$$\frac{d}{dt}\left(\mathcal{A}_c + \mathcal{A}_m\right) = 0\,, \tag{31}$$

denoting the mantle angular momentum by $\mathcal{A}_m$. However, the rapidly varying core surface motions that have been left out throughout the modelling remain mysterious.

There have been a few attempts to solve (28) numerically. In his pioneering study, Braginsky (1970) inferred the geometry of the meridional and quasi–static magnetic field from a plausible distribution of azimuthal electrical currents inside the core, making the condition (8) self–evident. His model included an insulating mantle and a conducting solid inner core, radius b, and angular velocity ω_i. He considered that the electromagnetic coupling between the torsional waves and the inner core is so efficient that

$$\forall\ s \leq b, \quad v_g(s) = s\omega_i \tag{32}$$

and solved (28) for $(v_g(s),\ s \geq b)$. Because, in his model, $\left\{B_s^2\right\}$ vanishes at $s = a$, the torsional oscillations are amplified in the equatorial region of the core. A few years ago, Buffett (1998) modified the Braginsky's model to include a possible gravitational torque between the inner core and the mantle and relinquished (32). I defer the discussion of this gravitational torque to the section (4) but Buffett considered also an electromagnetic torque acting on a conducting mantle, which interests us here. He relied on (28) simplified as it befits the axisymmetrical case. He noticed (his equation (32)) that

$$\frac{\partial\omega_g}{\partial s} = -\mu_0\sigma_m\Delta\frac{\partial\omega_g}{\partial t} \quad \text{at } s = a \tag{33}$$

when $B_r^2 \neq 0$ at the equator. Buffett used a numerical solution of the geodynamo equations (Kuang and Bloxham, 1999) as a substitute for a static basic state solution of (1). This does not, perhaps, represents an improvement on the initial state used by Braginsky since, very likely, the condition (8) is violated. As in the Braginsky's study and for the same reason, the oscillations are confined to the equatorial region; the boundary condition (33) does not constrain the solutions (fig. 1 of Braginsky (1970) and fig. 6 of Buffett (1998)), even for an insulating mantle, because $B_r^2\mid_{\theta=\pi/2} = 0$, in both models. The amplitude of the waves is chosen to match the characteristic amplitude of core surface motions inferred from s.v. data. Buffett found that a mantle conductance of 10^8 Siemens (S.) and a radial magnetic field at the CMB of uniform r.m.s. strength $5. \times 10^{-4}T$. would make the electromagnetic torque strong enough to couple the Alfvén torsional waves with the mantle and explain l.o.d. data. As Braginsky had forecast, this torque would be associated with heavy damping of the torsional waves. This modelling of the electromagnetic torque is however not conclusive because its success hinges on both a strong radial magnetic field B_r and a vanishing magnetic field B_s, in the equatorial region. The latter feature is required to enable amplification of the torsional waves, which augments the coupling. The two fields should merge at $s = a$ though.

Zatman and Bloxham (1997b; 1998) pioneered recently an inverse modelling of the torsional waves. As in core angular momentum studies, a model of $u_\phi(\theta, t)$, for 1900-90, has been transformed into a model of $v_g(s, t)$. The latter has then been converted into one or two torsional waves of definite imaginary frequency ϖ. In turn, the waves are inverted for models of $\left\{B_s^2\right\}$ and of an ad-hoc "friction" coefficient at the CMB. The model incorporates non–axisymmetrical effects. However, Zatman and Bloxham did not use the equation (28). Instead, they replaced $\tilde{\mathbf{b}}_m \mid_{r=a}$, the magnetic field at the bottom of the mantle, by $\tilde{\mathbf{b}} \mid_{r=a}$, the magnetic field induced in the core interior, in equation (28), and then they determined $\tilde{\mathbf{b}}$ by (15).That amounts to omit the magnetic diffusion layer. Zatman and Bloxham tried to get round this difficulty by introducing a coefficient α, $0 \leq \alpha \leq 1$ multiplying the surface term to mimic the influence of an insulating mantle, as found by Braginsky in the axisymmetrical case. But, as shown above, the surface term $J + J_\lambda$ does not vanish, even with an insulating mantle, in the non axisymmetrical case. Furthermore, in their model, α multiplies only a part of the surface term. The introduction of the coefficient α amounts indeed to replace $B_r \tilde{b}_\phi$ in the surface term by $(\alpha s B_s + z B_z)\, \tilde{b}_\phi / a$. Thus, taking $\alpha = 0$ does not suffice to cancel the surface term. It is not clear how the results depend on the incorrect substitution of $\tilde{\mathbf{b}} \mid_{r=a}$ for $\tilde{\mathbf{b}}_m \mid_{r=a}$. The most striking result is the abrupt increase of $\left\{B_s^2\right\}$ with colatitude θ at about $\theta = 60\,\text{deg}$.

Equation (28) gives also the response of the core to changes in the rotation rate of its container. Taking, as an example, the estimates of $\sigma_m \Delta$ and $B_r^2 \mid_{r=a}$ obtained from nutation studies, we calculate a spin-up time of the order of fifty years, reduced to 5 years for the outer geostrophic cylinders representing one tenth of core angular momentum. These values, which are probably on the lower side, are compatible with the observations reviewed in section (2).

I consider now non–axisymmetrical topography at the core–mantle boundary. Geostrophic contours are neither circular nor planar: any topography symmetrical with respect to the equatorial plane distorts the contours in the $s-$direction whilst antisymmetrical topography bends them in the $z-$direction. Yet, there are very few studies bearing on the coupled equations (12) and (13) when the contours are not circular. Anufriyev and Braginsky (1977) assumed that a zonal velocity $v_\phi\,(s, z, t)$, taken as representative of torsional waves, is present in an axisymmetrical reference state and investigated topographical effects as a perturbation only. They supposed also that the magnetic field $\mathbf{B}_M$, that is responsible for the torsional waves in the first place, can be neglected, as far as the perturbations caused by topography are concerned, in comparison with a zonal magnetic field $B_\phi(s,z)\mathbf{e}_\phi$. Their study aimed at evaluating the pressure torque that acts on the casing:

$$\int_\Sigma p\,(\mathbf{r}\times\mathbf{n})\,dS \tag{34}$$

In the event, topographical effects were found to be negligible. Braginsky (1998) followed up this study with another, in a plane layer approximation, including the consequence of a possible density stratification of the upper layers of the core. He found then that the pressure forces, at the CMB, exert a significant torque on the mantle. Topographical effects are indeed amplified because of the impenetrable boundary between the top layer and the core interior. The difficulty with the Anufriyev and Braginsky approach is its artificial character. In the actual problem, $\mathbf{v}_g$ flows along geostrophic contours, not along circular contours, and it is not perturbed by the aspherical boundary. Their study applies only if there is a core process, necessarily different from the torsional oscillations, that produce a zonal velocity $v_\phi\,(s,z,t)$ (with a decadal timescale) that does not follow the geostrophic contours. The work can be summarized as a study of forced Rossby waves in the presence of a magnetic field.

Unfortunately, the important point, i.e. the substitution of (12) to (14) when the contours are not circular has inspired very few studies, in the context of core–mantle coupling, at least. At first order ε in the topography:

$$r\mid_{cmb}= a\,(1+\varepsilon h\,(\theta,\phi))\,,\ h=O\,(1)\,. \tag{35}$$

There is now a contribution from the non–axisymmetrical part of the Lorentz force and from the buoyancy term. In the axisymmetrical case, multiplying (14) by s and integrating from $s=0$ to $s=a$ readily gives an expression of the torque acting on the core, which involves only the Lorentz force accelerating the rotation of the geostrophic cylinders. The situation is more intricate in the non–axisymmetrical case. Equation (12) does not yield a torque budget (Fearn and Proctor, 1992) and it does not indicate what forces exert a torque on the mantle. A somewhat artificial model incorporating an insulating mantle and a spherically symmetric gravity field is then an useful guide. Both gravity and electromagnetic torque vanish and yet the total core angular momentum, carried by geostrophic motions, can change. It turns out that, in this case, the pressure exerted on the CMB causes the angular momentum exchange between core and mantle even though the pressure gradient does not enter (12). Omitting entirely electromagnetic forces within the core and assuming $\varepsilon \ll 1$, Jault et al. (1996) found that the expressions giving respectively the action of the gravity force on each geostrophic cylinder $\mathcal{C}$ (equation 12) and the pressure torque exerted on the mantle at the rim of $\mathcal{C}$ are equivalent. In the general case (with magnetic forces), we have to rely on equation (12) to infer the time changes of the geostrophic velocity.

In conclusion, the theory of Alfvén torsional waves has given us, by far, the most robust link between theories of Earth's dynamo and observations. It explains most of the decade variations in the l.o.d. and, at least, some of the rapid variations of the Earth's magnetic field. Even if the first efforts of data assimilation have been promising, there is scope for further studies. An interesting step would be to construct a model of a quasi–static magnetic field satisfying either to (8) or to (5) and compatible with a model of the magnetic field at the Earth's surface. Through direct modelling, the hypotheses of topographic and electromagnetic coupling could then be tested. There is still no dynamical study of topographic coupling relevant to the Earth's core problem. In this context, the topography is important inasmuch it determines the geostrophic contours. It enters the model only through the equation (12) whilst magnetic field induction can be calculated in a spherical geometry. Concerning electromagnetic coupling, it is likely that it implies strongly damped torsional waves (Bloxham, 1998; Zatman and Bloxham, 1998). Relying on equation (28), this can be quantified. Finally, more sophisticated models of the geostrophic circulation inverted from magnetic field data and (31) are also possible. According to Zatman and Bloxham (1998), the decay times of the waves are at most of the order of their periods. As a result, the assumption that the geostrophic velocity, inside the core, can be represented as the superposition of a few waves with definite frequency appears questionable. This hypothesis is not necessary either since the frequency of the waves does not enter (28).

4 Deviations from axisymmetry of the solid inner core shape

If the two boundaries enclosing the fluid core, of radius respectively $r \simeq b$ and $r \simeq a$, are not perfectly axisymmetrical, there are regions void of geostrophic contours in the vicinity of $s = b$ and $s = a$. Near $s = b$, the crucial surface is the cylinder Π^b, which is parallel to the z-axis and touches the outer rim of the inner core, at a location denoted $z^b(\phi)$. The cylinder intersects the outer boundary at $z = z^b_T(\phi)$ and $z = z^b_B(\phi)$. Just inside Π^b and above the inner core, there is a geostrophic cylinder $C\left(H^{b,i}\right)$ of constant height $H^{b,i} = \min\left(z^b_T(\phi) - z^b(\phi)\right)$ but there are no geostrophic contours between $C\left(H^b\right)$ and Π^b. In the same way, there is a geostrophic cylinder $C\left(H^{b,o}\right)$ of constant height $H^{b,o} = \min\left(z^b_T(\phi) - z^b_B(\phi)\right)$ just outside Π^b but no geostrophic contours between $\Pi^b$and $C\left(H^{b,o}\right)$. In the absence of closed contours of constant height, the role of the geostrophic motions is usually taken over by low frequency z–independent inertial waves (Greenspan, 1968), known as "Rossby waves". Outside Π^b, the height variation of fluid columns circling around the inner core is of the order of h/a and the frequency of the Rossby waves is thus of the order of $(h/a)\,\Omega$. This is comparable with the frequency of the torsional Alfvén waves. On the other hand, the special regions, where no geostrophic contours exist, represent a small portion of the fluid volume. This explains that they do not play a role in the model of Buffett (1996), which I outline now.if

Buffett studied the coupling of the rotation of the inner core to the torsional waves in the fluid outer core. He supposed that the inner core surface Σ_b is an equipotential surface of the Earth's gravity field. The hydrostatic pressure is indeed constant, in the fluid outer core, on these equipotential surfaces and the pressure determines the freezing point of iron. Because of density anomalies in the mantle, the Earth's gravity field is probably not axisymmetrical and neither is the inner core surface (see the Introduction). Buffett studied the free mode of axial rotation of the solid inner core in this configuration. When the inner core is rotated from its equilibrium position, an archimedean force arises in response to the misalignment between the inner core and the mantle. The gravity torque acting on the fluid outer core from the mantle is compensated by the pressure torque acting from the inner core whilst the net torque acting on the inner core is non zero because of the density jump at Σ_b. Finally, Buffett found that the period of this rotation eigenmode is of the order of a few years. This is small compared to the period of the torsional waves. Hence, according to this study, the inner core is locked to the mantle. In addition to all the other torques acting on the mantle, the fluid outer core and the mantle may thus be coupled through the electromagnetic torque acting between the fluid and solid cores. This mechanism can theoretically be tested from seismological observations of the inner core rotation (Buffett and Creager, 1999). Viscous deformation of the inner core in response to gravity and pressure forces may loosen the grip of the mantle on the inner core. Assuming a newtonian rheology for the inner core (neglecting elastic deformation), Buffett investigated different values of the viscous relaxation time of the inner core τ_v. He concluded, using the rather high value of inner core topography found by Forte and Peltier (1991), that gravitational coupling between the inner core and the mantle remains important within a wide range of values of τ_v. This mechanism involves electromagnetic coupling between the inner and outer cores and thus implies some damping of the torsional waves (Buffett (1996)). Finally, if the gravitational torque is important, the ensemble inner core - mantle is coupled to the fluid core through torsional waves of period τ_{TA}. This is once more compatible with the results of section 2 about the characteristic timescale of the coupling mechanism between outer core and mantle. Gravitational coupling is an attractive mechanism to explain changes in l.o.d. Assessment of its importance awaits renewed confrontation with geomagnetic and seismic data (see the promising study of Vidale et al. (2000)) together with improved modelling of topographic and electromagnetic coupling at the CMB.

5 Kinematic modelling

Considering core–mantle coupling as a consequence of torsional Alfvén waves, we presume that we have attained a good understanding of core dynamics. However, torsional waves do not explain all the rapid changes of the Earth's magnetic field. Furthermore, I conclude from the above review that topographic and electromagnetic coupling have not yet been satisfactorily incorporated in models of torsional waves. These weaknesses of a fully dynamic approach justify the less ambitious kinematic studies that I report now. Neglecting diffusion, models of core surface motions $\mathbf{u}\mid_{r=a}$ have been inverted from the radial component of the induction equation

$$\frac{\partial B_r}{\partial t} = -\nabla_H \cdot (\mathbf{u}B_r) \ . \tag{36}$$

They include large scale, non geostrophic flows that change on a decadal timescale. In the present state of core studies, these motions appear enigmatic. Leaving aside the question of their origin, a kinematic modelling of core–mantle coupling is nevertheless possible. It consists in investigating the different torques acting on the mantle that are associated with these flows. In this context, two mechanisms have been particularly studied. First, electrical potential differences are set up at the core surface. Thus, electrical currents flow in the mantle if it is not a perfect insulator and a Lorentz force exerts a torque on the mantle. Second, a pressure is associated to the motions and is applied also on the solid mantle.

Roberts (1972) long ago gave the general formulation of the electromagnetic torque. The approximate expression derived in section 3 applies to the case of zonal motions at the core surface and thin electrically conducting layer at the bottom of the mantle but the conclusions are not radically altered in less specific cases. All authors have assumed that the magnetic field induced in the mantle is a perturbation of the magnetic field generated by core motions (Benton and Whaler, 1983). In other words, the electrical potential differences are not short-circuited by the conducting mantle. It is convenient to write again the continuity of the electrical field tangent to the boundary across this surface. The motions $\mathbf{u}\mid_{r=a}$ enter the equation (36) only through the term:

$$\mathbf{u}B_r = \nabla_H \Psi + \nabla \times (\mathbf{r}\Phi)\,, \quad V = a\Phi\ , \tag{37}$$

where V is electrical potential and Ψ and Φ are two scalar fields on the core surface. The electrical potential can be calculated in the mantle from its value at the CMB and the equation for electrical charge conservation:

$$\nabla \cdot (\sigma_m \nabla V) = 0. \tag{38}$$

From (36), the calculation of Ψ is straightforward. This term arises because of time changes of the magnetic field permeating the conducting mantle; it is associated with electrical currents that can be directly calculated from s.v. models also. However, the magnetic field, in the mantle, is largely axisymmetrical. Taking into account $\mathbf{B}_M(r,\theta)$ only, the azimuthal electrical currents induced by the time changes of $\mathbf{B}_M(r,\theta)$ do not enter the expression of the azimuthal Lorentz force $F_{B,\phi} = (\mathbf{j}\times\mathbf{B})_\phi$ that acts as a torque on the mantle. On the other hand, an electrical potential $V(a,\theta)$ is set up by any zonal motion at the core surface (see equation 37). The resulting magnetic force in the mantle,

$$\mathbf{F}_B = \sigma_m \nabla V(r,\theta) \times \mathbf{B}_M(r,\theta)\,, \tag{39}$$

is directed along $\mathbf{e}_\phi$ and is very efficient to torque the mantle. Finally, the main part of the torque is determined by the unknown scalar Φ , which is almost not constrained by s.v. data (Jault and Le Mouël, 1991) because the magnetic field in the mantle is predominantly zonal. Holme (1998a) relied on this property to show, through inverse modelling, that there are models of $\mathbf{u}\,|_{r=a}$ that are compatible with geomagnetic observations and yield an electromagnetic torque that, taken in isolation, would explain l.o.d. data, if the mantle is sufficiently conducting. Using annual means of the magnetic field published by the observatories complemented by the Bloxham and Jackson (1992) magnetic field model for the period 1900-1980, he found $\sigma_m \Delta = 10^8 S$ as a reasonable minimum value for the mantle conductance (Holme, 1998b) to make the electromagnetic torque significant. This is about 5 times less than the value that was deemed necessary from direct modelling. Wicht and Jault (1999) based their investigation of the electromagnetic torque on Φ instead of $\mathbf{u}\,|_{r=a}$, as in Holme (1998a). The idea was to monitor precisely the uncertainty. There is indeed one single information on Φ :

$$\nabla \times (\mathbf{r}\Phi) = -\nabla_H \Psi \quad \text{when} \quad B_r = 0\,. \tag{40}$$

Knowing that we are interested only by Φ, we tried to give more weight to (40). However, we mainly confirmed the result of Holme (1998b) for the minimum value of $\sigma_m \Delta$ and we were not able to increase it despite the refined constraint (40). Finally, the consequences of possible lateral variations in the electrical conductivity at the bottom of the mantle have been recently investigated (Holme, 2000; Wicht and Jault, 2000). It turns out that the contributions of the different regions at the core surface are simply weighted by the conductance of the mantle nearby.

If the core–mantle boundary is aspherical (see equ. (35) defining the small parameter ε), the moment of the pressure force acting on the mantle from the core may be non zero (Hide (1969); see equation (34)). Neglecting the Lorentz force at the core surface, a kinematic approach is possible (Hide, 1989; Jault and Le Mouël, 1989). The pressure is calculated in the spherical approximation:

$$\mathbf{u} = \mathbf{u}_0 + \varepsilon \mathbf{u}_1; \quad p = p_0 + \varepsilon p_1 \tag{41}$$

At the core surface, and at zeroth order in ε, the equations (1) and (3) are transformed into:

$$2\rho\,(\Omega \times \mathbf{u}_0) = -\nabla p_0 - \alpha\theta\mathbf{g}, \quad \mathbf{u}_0 \cdot \mathbf{e}_r = 0. \tag{42}$$

According to the kinematic approach that I adopt here, the velocity $\mathbf{u}_0$ is known from (36). The pressure p_0 is then obtained from the horizontal components of (42). The pressure torque arises at first order in ε. Omitting other possible torques for the sake of simplicity (they can be reinstated later) and operating with $\int \mathbf{r}\times$ on the equation of motion, I obtain:

$$\int \mathbf{e}_\phi . s\rho \frac{\partial \mathbf{u}_0}{\partial t} dV = -\mathbf{e_z} . \left(\int_{cmb} (\mathbf{r} \times p_0 \mathbf{n})\, d\Sigma + \int_{r=a} (\mathbf{r} \times p_1 \mathbf{r})\, d\Sigma \right) \tag{43}$$

The contribution of the first-order pressure vanishes and a model of p_0 , together with a model of CMB topography, suffices to calculate the pressure torque. In order to make the derivation consistent, the inertial term has to enter the equation of motion at the order ε also. This means that the angular momentum carried by core motions changes on the characteristic time $(\varepsilon\Omega)^{-1}$. Decadal timescales are obtained for topographies of a few hundreds of meters. More detailed studies are frustrated by the lack of models of the topography at the CMB. In addition, the pressure force with short lengthscale along the core surface is not constrained by s.v. data but may nevertheless exert a significant torque on the mantle. These two shortcomings explain that there have been no attempts at calculating a time series of the pressure torque from s.v. data. If anything, the topographic torque inferred from kinematic modelling is too potent.

6 Concluding remarks

As we have seen, the main limitation of the models of torsional Alfvén waves is their inability to explain some rapid changes in the Earth's magnetic field that are currently attributed to non–geostrophic large scale motions, themselves of unknown origin. It is, of course, possible to devise explanations for such global flows. Braginsky (1993) noticed that if the upper core were chemically stratified, it would support fast waves of long lengthscale. Another plausible way to reconcile the theory of torsional waves with observations is to interpret the rapid variations of the Earth's magnetic field as the result of short lengthscale motions. As a matter of fact, an important part of the s.v. models can already be well explained by steady or slowly varying large scale core surface motions, which are expected from dynamo modelling. Rapidly varying flows are often penalized in models of velocity at the core surface, but short and long lengthscale motions are treated in the same way whilst rapid changes of the short lengthscale component are much more acceptable from a physical standpoint. Note also that the small scale components, of harmonic degree $l \geq 10$, have been underestimated in the magnetic field model that has been the reference for the last 15 years (Bloxham and Jackson, 1992). This has made the contribution of small scale motions to the secular variation potentially less important. Finally, a temporal norm has been minimized in the inversion of the magnetic field model itself. It does not distinguish either between short and long lengthscale components. All these assumptions may well have conspired to mistake rapid variations of the small scale flow for large scale motions. Before the satellite era that we are now entering, it was difficult to test more sophisticated a priori models of the magnetic field and of the surface core flows for want of data. Now, observations of geomagnetic field, with smaller lengthscales and shorter timescales become available. Thus, the modelling of core surface motions will require less regularization.

Finally, there is an interesting theoretical question still pending. It transpires that the amplitude of the torsional wave velocity matches up to the amplitude of the quasi–static velocity field, which is part of the dynamo process. This coincidence calls for an explanation, which may involve a discussion of the coupling with the mantle. In the view expounded here, excitation of the Alfvén torsional waves stems from the slow evolution of the quasi–static state. On the other hand, the coupling mechanism between the core and the mantle is not immaterial to the amplitude of the torsional waves since electromagnetic coupling with either the inner core gravitationally locked to the mantle or directly with the mantle probably entails strong damping. The situation is different in the case of topographic coupling.

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