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Supplementary Material for "Dynamic regimes in planetary cores: $\tau - \ell$ diagrams"

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This article is a draft (not yet accepted!)

S1. τ - ℓ diagram for convection onset

Although $\tau - \ell$ regime diagrams are built to span a large range of length- and time-scales, they provide an interesting insight on what controls single scales appearing at the onset of convection. Here we compare what would be the convection threshold in Earth's core, depending on whether it is rotating or not. We display the results in Figure S1, using properties of the Earth's core listed in the article's Table II. We extended the figure to very long τ -values in order to include the intersection of the viscous line and right *y*-axis.

In the absence of rotation, the threshold of convection is governed by a balance between buoyancy and the combined action of momentum and thermal diffusions. It takes place at the largest length-scale $\ell = R_o$ (or at 'scale height' $\mathcal{H} = \frac{C_p}{ag}$ if it is smaller than R_o), and for $\operatorname{Ra}(R_o) \sim 1$ (note that critical value Ra_c is in fact much larger than 1 due to several powers of 2π , which we dropped for simplicity). Expressing ℓ -scale Rayleigh number as $\operatorname{Ra}(\ell) = \frac{\tau_\kappa(\ell)\tau_\nu(\ell)}{\tau_\rho^2(\ell)}$, following the article's Table III, we obtain critical τ_ρ at the convection onset: $\tau_{\rho_c} = \sqrt{\tau_\kappa(R_o)\tau_\nu(R_o)}$. This value is plotted in Figure S1 as an orange disk on right *y*-axis, at a time half-way between $\tau_\kappa(R_o)$ and $\tau_\nu(R_o)$.

Things get very different when the system is rapidly rotating. Proudman-Taylor constraint inhibits convective flows, and viscosity is needed to break this geostrophic constraint [Chandrasekhar, 1961]. Convection marginal stability in rapidly rotating spheres has a long history [Chandrasekhar, 1961, Roberts, 1968, Busse, 1970, Jones et al., 2000, Zhang et al., 2007]. Convective structures are found to be quasi-geostrophic at onset, forming columnar vortices with a width small enough to enable viscosity in the bulk to alleviate Proudman-Taylor constraint, yielding the famous $Ek^{1/3}$ law. These structures can also be viewed as thermal Rossby waves.

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Figure S1. $\tau \cdot \ell$ diagram illustrating convection onset. In the absence of rotation, convection sets in with a cell size comparable with R_o . Orange disk marks the value of τ_ρ at convection onset. It is half-way between $\tau_\kappa(R_o)$ and $\tau_\nu(R_o)$, where $\operatorname{Ra}(R_o) \sim 1$. Rotation controls the length-scale at which convection sets in. We compare a $\operatorname{Pr} < 1$ case with a $\operatorname{Pr} > 1$ case. Black down-triangle marks column radius ℓ_c and period τ_c of the quasi-geostrophic vortices that form at convection onset when considering viscosity and thermal diffusivity ($\operatorname{Pr} < 1$). Orange down-triangle at $\ell = R_o$ indicates the value of τ_ρ at onset. Up-triangles display the same quantities when considering viscosity and compositional diffusivity χ ($\operatorname{Pr} > 1$). Dotted lines help reading their graphical construction.

It is interesting to examine the graphical $\tau - \ell$ representation of this situation. We follow the local stability analysis of Busse [1970]. Omitting numerical prefactors, his equations (4.11) to (4.13) yield:

$$\ell_c \simeq R_0 \left(\Pr^{-1} + 1 \right)^{1/3} \text{Ek}^{1/3} \tag{1}$$

$$\tau_c \simeq t_\Omega \Pr^{1/3} (1 + \Pr)^{2/3} E k^{-1/3}$$
(2)

$$\tau_{\rho_c} \simeq t_{\Omega} \left(\Pr^{-1} + 1 \right)^{2/3} \Pr^{1/2} E k^{-1/3}, \tag{3}$$

where ℓ_c is column radius, τ_c time period, τ_{ρ_c} free-fall time (for $\ell = R_o$) at onset, and $Pr = \nu/\kappa$ the Prandtl number. Note that Ek is here the classical large-scale Ekman number $Ek(R_o)$.

Figure S1 translates these results graphically. At convection onset, column width is at the intersection of line τ_{Rossby} and either line τ_v or line τ_κ line, depending on which it encounters first. Thermal (or compositional) diffusivity governs period since $\tau_c \simeq \tau_\kappa(\ell_c)$ in both cases. We also observe that $\tau_{\rho_c} = \sqrt{\tau_v(\ell_c)\tau_\kappa(\ell_c)}$. Graphically, this places τ_{ρ_c} along right *y*-axis ($\ell = R_o$) at a time half-way between $\tau_v(\ell_c)$ and $\tau_\kappa(\ell_c)$ on our log-log plots. Note that onset parameters from global stability analysis [Jones et al., 2000, Dormy et al., 2004] differ substantially from those of the local stability analysis of Busse [1970] for Pr < 1.

S2. τ - ℓ regime diagram of numerical dynamo simulations

We have shown $\tau - \ell$ diagrams computed from numerical simulations in sections 4 and 5 of the article. We complement these with two numerical simulations proposed by Dormy [2016]. Figure S2 displays $\tau - \ell$ regime diagrams of a 'weak' and 'strong' dynamo at large magnetic Prandtl number Pm.



Figure S2. $\tau - \ell$ regime diagrams of numerical simulations reproducing 'weak' and 'strong' dynamo branches of Dormy [2016]. Same conventions as in earlier Figures. Horizontal dashed lines indicate power dissipation times. The viscous one is pinned to line $\tau_{\nu}(\ell)$, while Ohmic dissipation time is pinned to line $\tau_{\eta}(\ell)$. Their proxies approximated by equation (50) of the article's Appendix A are shown as horizontal dotted lines.

Their parameters are taken close to those given in lines 1 and 2 of Table 1 of Dormy [2016]: $Ek(R_o - R_i) = 3 \times 10^{-4}$, Pr = 1 and Pm = 18. A $Ra^*(R_o)$ value of 3.344×10^5 yields a weak-field dynamo (Figure S2a), while a strong-field dynamo is obtained for $Ra^*(R_o) = 3.648 \times 10^5$ (Figure S2b). While lines $\tau_u(\ell)$ and $\tau_\rho(\ell)$ of the two dynamos do not differ much, the strong dynamo has much larger magnetic intensity (lower line $\tau_b(\ell)$) and Ohmic dissipation (lower horizontal red dashed line). Both dynamos appear close to or within the viscous dissipation region. Lines $\tau_u(\ell)$ of both dynamos are very similar. They plot above the Rossby line but largely invade the viscous dissipation range, left of line $\tau_v(\ell)$, and above spin-up time.

Dormy [2016] observes the co-existence of weak and strong dynamo branches in some Rayleigh number range. We note that this co-existence would disappear if power input rather than Rayleigh number were chosen as control parameter. Indeed, the two simulations mainly differ by the magnetic field intensity they produce (as expected) and by the amount of dissipation. While viscous dissipation keeps a similar level, total dissipation is almost three times larger in the strong dynamo, due to a 40 fold increase of Ohmic dissipation. In other words, magnetic field is weak on the weak branch because it's all what dynamo action can produce given available convective power, while it gets strong as soon as enough power is supplied, which is not fixed in Dormy's setup.

S3. τ - ℓ regime diagram of the DTS liquid sodium experiment

We think that laboratory experiments can also provide a better perspective when translated into $\tau - \ell$ regime diagrams. As an illustration, we present in Figure S3 the composite $\tau - \ell$ diagram of a representative run of

DTS experiment. DTS experiment is a magnetized spherical Couette experiment. Fifty liters of liquid sodium are enclosed in a spherical container ($R_o = 0.21$ m) that can rotate around a vertical axis. A central inner sphere ($R_i = 0.074$ m) can rotate independently around the same axis, and hosts a strong permanent magnet producing an axial dipolar magnetic field. See Nataf et al. [2008], Brito et al. [2011] for more details.



Figure S3. Composite representative $\tau \cdot \ell$ diagram of DTS magnetized spherical Couette laboratory experiment. Outer shell rotation frequency is 10 Hz, while inner sphere spins at 20 Hz. Time t_{Alfven} (large red dot) is obtained from the energy of the dipolar magnetic field applied by the inner sphere magnet, while time t_{b0} (small red dot) represents time-averaged induced magnetic field. Time t_{SV} (blue dot) is deduced from time-averaged kinetic energy. Red horizontal dotted line marks dissipated power (700 W). Thick red line $\tau_b(\ell)$ is obtained from a *k*-spectrum of magnetic fluctuations measured at the outer shell surface. Thick blue line $\tau_u(\ell)$ is obtained from a frequency-spectrum of flow velocity fluctuations measured at mid-depth in the fluid using Ultrasound Doppler Velocimetry.

We draw τ_v and τ_η lines from properties of liquid sodium at 130 ° C ($v = 6.5 \times 10^{-7} \text{ m}^2 \text{s}^{-1}$, $\eta = 8.8 \times 10^{-2} \text{ m}^2 \text{s}^{-1}$). Power markers along these lines are deduced from liquid volume and density ($\rho = 930 \text{ kgm}^{-3}$). From the energy of applied dipolar magnetic field, we deduce time t_{Alfven} and draw the $\tau_{Alfven}(\ell)$ red wavy line of Alfvén wave propagation. We pick a run with outer shell rotation rate $f_o = 10 \text{ Hz}$, which yields horizontal line t_{Ω} and wavy line τ_{Rossbv} .

We consider an inner sphere differential rotation rate $\Delta f = 10$ Hz, for which a power dissipation $\mathcal{P}_{diss} \simeq 700$ W is measured. Cabanes et al. [2014] reconstructed time-averaged flow and induced magnetic field by a joint inversion of a comprehensive set of flow velocity profiles and magnetic field measurements. Their Table III provides the energies of these fields, which we convert into vortex overturn time at integral scale t_{SV} , and the corresponding time for induced magnetic field t_{b0} . Note that their analysis if for $f_o = 0$, but should approximatively apply to our case.

The three components of the induced magnetic vector are measured every 6° along a meridian between latitudes -57° and +57°. We thus computed a *k*-spectrum of magnetic energy density from a set of 60s-long

records, which was converted into line $\tau_b(\ell)$ according to equation (34) of the article's Appendix A.

No velocity measurement was available for that run, but a nice profile of angular velocity was measured using Ultrasound Doppler Velocimetry for another run with $f_o = 5$ Hz and $\Delta f = 10$ Hz. Using equation (47), we extract the frequency power spectral density of a 40s-long record at fluid mid-depth, and convert it to a kinetic energy density *k*-spectrum using equation (48) of the article's Appendix A, with *U* deduced from the same profile, yielding line $\tau_u(\ell)$ drawn in Figure S3. Note that this spectrum might be contaminated by instrumental noise.

The resulting $\tau - \ell$ diagram suggests that velocity fluctuations are mostly quasi-geostrophic because line τ_u plots above Rossby line. Magnetic energy fluctuations are in a strongly dissipative region, and are almost three orders of magnitude smaller that kinetic energy fluctuations, in agreement with observations of Figueroa et al. [2013]. Under the combined influence of strong rotation and strong imposed magnetic field, energy fluctuations of both types are two to three orders of magnitude smaller than time-averaged energies (tagged by t_{SV} and t_{b0} in Figure S3), as noted by Nataf and Gagnière [2008], Kaplan et al. [2018].

The short red wavy line $\tau_{Alfven}(\ell)$ indicates that geostrophic Alfvén waves can propagate but are severely damped, as analyzed by Tigrine et al. [2019]. Dissipation is dominated by Ohmic dissipation of the time-averaged flow.

S4. τ - ℓ Python package

In the attached tau-ell_programs.zip archive, we provide short Python scripts used to draw our article's figures.

- tau_ell_lib.py: library gathering τ - ℓ conversion rules from spectra, and graphical functions.
- read_parameters.py: document the parameters of the natural object, or numerical simulation, or Lab experiment, for which one wishes to draw $\tau \ell$ regime diagrams (or templates).
- plot_Kolmogorov.py: plot τ - ℓ diagram of Kolmogorov's universal turbulence.
- plot_rotating_convection.py: plot $\tau \ell$ diagram of rotating convection in QG-CIA force balance.
- plot_core_dynamo.py: plot *τ*-ℓ diagram of rotating convective dynamo (either MAC, QG-MAC or QG-MAC_JA force balances)
- plot_DNS_dynamo.py: plot τ - ℓ diagrams of numerical simulations, given their spherical harmonic degree *n*-spectra.
- plot_Lab_experiment.py: plot *τ*-ℓ diagrams of Lab experiments, given their frequency of wavenumber spectra.
- plot_convection_onset.py: plot dynamo onset parameters for rotating and non-rotating convection.

The numerical simulation and Lab experiment data used for the examples shown in our article are available in folders:

- DNS_Guervilly: *u* and ρ spectra of Guervilly et al. [2019]'s 3D rotating convection simulation at Ek = 10^{-8} .
- DNS_Schaeffer: u, b, and ρ average spectra of Schaeffer et al. [2017]'s S2 numerical geodynamo simulation.
- DNS_Dormy: *u*, *b*, and *ρ* average spectra of weak and strong dynamo numerical simulations proposed by Dormy [2016].
- experiment_DTS: *u* frequency spectrum, and *b* wavenumber spectrum of a composite run of the DTS liquid sodium experiment.

This package is also available at https://gricad-gitlab.univ-grenoble-alpes.fr/natafh/tau-ell_programs.

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