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Discontinuous Galerkin Method for TTI Eikonal Equation

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Summary

A new formulation for solving the Eikonal equation is investigated using a time-dependent Hamilton-Jacobi equation. A discontinuous Galerkin (DG) finite element method is proposed for direct reconstruction of the traveltime field as the final stationary solution. Both isotropic and tilted transversely isotropic (TTI) implementations are performed in heterogeneous media with stable and accurate results as long as we honor the high-frequency approximation. We introduce outgoing conditions at edges able to handle complex topography and we deal with singularity at the source through the additive factorization. Expected convergence behavior regarding element interpolation is observed when considering factorization. Comparison between DG and finite difference solutions in the complex BP TTI model with unstructured and structured meshes illustrates the highly accurate traveltime estimation of this DG approach, pointing out perspectives for integrating this accurate local Eikonal solver into efficient methods for getting the stationary solution, such as fast sweeping methods.

Introduction

Traveltime computation is of major interest for a large range of applications in seismic processing and imaging, among which traveltime tomography for velocity macro-model building before subsequent Full Waveform Inversion (FWI) / Migration Velocity Analysis (MVA), phase identification and data windowing. While Lagrangian ray-tracing algorithms remain efficient tools when the velocity field is sufficiently smooth, they are no longer suitable for heterogeneous media where the regular sampling of the medium becomes a challenge with triplications and shadow zones, especially when a first-arrival traveltime is required everywhere in the medium. Since the work of Vidale (1990), many Eikonal solvers have been developed, most of them in a finite-difference framework (*e.g.* Podvin and Lecomte (1991); Noble et al. (2014)). The robustness of such Eulerian schemes for first-arrival traveltime computation relies on the definition of viscosity solutions given by Crandall and Lions (1983) for Hamilton-Jacobi equations, a general class to which Eikonal equation belongs. Yet first-arrival traveltime computation in strongly heterogeneous media is still challenging due to the low-order convergence of finite-difference schemes in comparison with the high level of precision with which traveltimes and their derivatives need to be estimated for applications such as beam migration or slope tomography: intense numerical developments are still undertaken. In addition, anisotropy has become a vital tool for medium description in the recent developments of inversion strategies, so that recent works focus on building anisotropic solvers (Waheed et al., 2015; Tavakoli F. et al., 2015). We propose a novel strategy for computing traveltimes in anisotropic media by introducing a discontinuous Galerkin (DG) scheme from Cheng and Wang (2014) able to handle a direct Hamiltonian formulation for the Eikonal equation. We exhibit numerical convergence rates on a simple model and we present an illustrative example on the BP TTI benchmark model before we conclude on potentialities of this approach.

Theory: Hamiltonian formulation and Eikonal factorization

We focus on 2D geometries, but the equations we present can be directly extended to 3D. We introduce a pseudo-time t and we solve the time-dependent Hamilton-Jacobi equation

$$\frac{\partial u(\mathbf{x},t)}{\partial t} + \mathscr{H}(\mathbf{x}, \nabla_{\mathbf{x}} u(\mathbf{x},t)) = 0 \tag{1}$$

for $u(\mathbf{x},t)$ where $\mathbf{x}$ denotes the spatial coordinates. The steady state of $u(\mathbf{x},t)$ corresponds to the solution of the Eikonal equation (traveltimes). For a velocity field $c(\mathbf{x})$, one possible Hamiltonian function for the isotropic case is

$$\mathscr{H}_{\text{ISO}}(\mathbf{x}, \nabla_{\mathbf{x}} u(\mathbf{x},t)) = \|\nabla_{\mathbf{x}} u(\mathbf{x},t)\| - \frac{1}{c(\mathbf{x})}. \tag{2}$$

Vertical transversely isotropic (VTI) Hamiltonian function, following Slawinski (2003) with the use of Christoffel's dispersion relation and Thomsen's parameters (Thomsen, 1986), is defined by

$$\mathscr{H}_{\text{VTI}}(\mathbf{x}, \nabla_{\mathbf{x}} u(\mathbf{x},t)) = A u_{,z}^4 + B u_{,x}^4 + C u_{,x}^2 u_{,z}^2 + D u_{,x}^2 + E u_{,z}^2 - 1, \tag{3}$$

with $A = -V_p^2 V_s^2$, $B = -(1+2\varepsilon)V_p^2 V_s^2$, $C = -(1+2\varepsilon)V_p^4 - V_s^4 + (V_p^2 - V_s^2)\left[V_p^2(1+2\delta) - V_s^2\right]$, $D = \left[V_s^2 + (1+2\varepsilon)V_p^2\right]$, $E = (V_p^2 + V_s^2)$. Indices $_{,z}$ and $_{,x}$ indicate the derivatives with respect to z and x. From this VTI formulation, the TTI case is obtained by the local rotation

$$u_{,x} \to u_{,x}\cos\theta + u_{,z}\sin\theta, \quad u_{,z} \to u_{,z}\cos\theta - u_{,x}\sin\theta, \tag{4}$$

where the tilt angle $\theta(\mathbf{x})$ denotes the local angle made by the symmetry axis with the vertical. We may simplify the expression (3) by setting $V_s = 0$. This acoustic approximation does not perturb the numerical solution when considering only first-arrival traveltimes (Alkhalifah, 2000).
We then use the factorization method in the case of a point source (Fomel et al., 2009), in order to mitigate the numerical error due to the source singularity by considering the additive decomposition

$$u(\mathbf{x},t) = u_0(\mathbf{x}) + \tau(\mathbf{x},t). \tag{5}$$

The reference solution u_0 is known for an homogeneous medium with the velocity value taken at the source. We obtain a Hamilton-Jacobi equation for the new field $\tau(\mathbf{x},t)$ by inserting expression (5) into equations (1) to (3). The resulting equation is solved by the DG method we introduce in the following.

Numerical approach: Discontinuous Galerkin method

Recently, Cheng and Wang (2014) proposed a DG scheme that simplifies the numerical implementation for computing the field $\tau(\mathbf{x},t)$. The spatial domain Ω is partitioned into n polygonal elements

denoted by K_i, $i = 1,...,n$, using structured or unstructured meshes (h-adaptivity). A local approximation space $\mathscr{P}_i$ of dimension d_i is chosen for each element K_i together with a basis of shape functions $\phi_i^j(x,z)$, $j \in \{1,...,d_i\}$. The choice of the approximation space may differ from one element to another (p-adaptivity). Here, we use classical polynomial approximations of degrees from 1 to 3, denoted by P^1, P^2 and P^3. At the steady state, traveltimes are approached in a weak sense by the numerical solution u_h inside each element. We define $\mathbf{n}_{K_i}$ to be the outward unit normal to the K_i cell boundary. At cell interfaces, traces $v_h^{\pm}$, jumps $[v_h]$ and means $\overline{v_h}$, of any numerical quantity v_h defined inside two neighboring cells are given respectively by

$$v_h^{\pm}(\mathbf{x}) = \lim_{\varepsilon \downarrow 0} v_h(\mathbf{x} \pm \varepsilon \mathbf{n}_{K_i}), \quad [v_h](\mathbf{x}) = v_h^+(\mathbf{x}) - v_h^-(\mathbf{x}), \quad \overline{v_h}(\mathbf{x}) = \frac{1}{2}\big(v_h^+(\mathbf{x}) + v_h^-(\mathbf{x})\big). \tag{6}$$

With these expressions, as well as with their spatial derivatives and their projections on boundaries, different key quantities $\mathscr{F}, \mathscr{G}, \mathscr{K}$ are defined at the boundary between two elements in order to build a causality consistent flux estimation, which is an essential ingredient of the DG formulation. For the sake of concision, the expressions for $\mathscr{F}, \mathscr{G}, \mathscr{K}$ are not reproduced here: let us mention these quantities depend nonlinearly on the solution itself and on the local values of the Hamiltonian function (Cheng and Wang, 2014).

Inside the discretized space where ΔK_i is the K_i element size and ΔS_i^j the length of edge j of element K_i, the weak formulation of the equation (1) can be expressed as

"Find the discrete field $u_h(.,t) \in \{v : v|_{K_i} \in \mathscr{P}_i, \forall i \in \{1,...,n\}\} \forall t > 0$, such that

$$\begin{aligned} &\int_{K_i} \Big(\partial_t u_h(\mathbf{x},t) + \mathscr{H}\big(\nabla_{\mathbf{x}} u_h(\mathbf{x},t), \mathbf{x}\big)\Big) v_i(\mathbf{x}) \mathrm{d}\mathbf{x} \quad + \quad \int_{\partial K_i} \mathscr{F}\,[u_h](\mathbf{x},t) v_i^-(\mathbf{x}) \mathrm{ds} \\ &- C\Delta K_i \left(\sum_{S_i^j \in \partial K_i} \frac{1}{\Delta S_i^j} \int_{S_i^j} \mathscr{G}\,[\nabla_{\mathbf{x}} u_h \cdot \mathbf{n}_{K_i}](\mathbf{x},t) v_i^-(\mathbf{x}) \mathrm{ds} \; + \sum_{\bar{S}_i^j \in \bar{\partial} K_i} \frac{2}{\Delta \bar{S}_i^j} \int_{\bar{S}_i^j} \mathscr{K}\,\left(\nabla_{\mathbf{x}} u_h^-(\mathbf{x},t) \cdot \mathbf{n}_{K_i}\right) v_h^-(\mathbf{x})\, \mathrm{ds} \right) = 0, \end{aligned} \tag{7}$$

for each $i \in \{1,...,n\}$ and for any test function $v_i \in \mathscr{P}_i$."

The set ∂K_i denotes the internal edges which are shared with other cells, and $\bar{\partial} K_i$ the external edges which are parts of the domain boundary $\partial\Omega$. The test functions v_i are shape functions as usual for Galerkin approaches (Zienkewicz and Morgan, 1983). The three boundary integrals involving $\mathscr{F}$, $\mathscr{G}$ and $\mathscr{K}$ rule the interaction between elements. The first term of the scheme (7) embeds the Hamilton-Jacobi equation for consistency. The second term determines the information flow direction and allows shock capture when present. The third term treats the so-called rarefaction situation and is balanced by the viscosity parameter C set to 0.25 as a good empirical value (Cheng and Wang, 2014). We add the last new term for handling outgoing-flux boundary conditions. The time integration of the equation (7) is performed by a second-order Runge-Kutta scheme until we reach the steady-state solution. We emphasize that the scheme (7) holds for isotropic as well as for TTI cases, which highlights the flexibility of this approach.

Numerical examples

We first perform a convergence study on a simple isotropic model which consists of a constant vertical velocity gradient. An exact analytical solution is known for such a model (Červený, 2001). We consider a square domain of 4000 m length along x- and z-axes with a point source located at the center. Inside this domain, we define a linear dependency of the velocity along z-coordinate, which varies from 1000 ms^{-1} at $z = 0$ to 3000 ms^{-1} at $z = 4000$ m. The domain is firstly discretized in a rectangular Cartesian way with $N_x = N_z = N$ elements along x- and z-axes so that the total number of elements is N^2. Another discretization is obtained by cutting each rectangular element into two triangles in an alternating diagonal direction. The number of elements of this so-called Union-Jack (UJ) pattern is now $2N^2$. We perform computation for both settings. Based on theoretical results, we expect at most k+1-convergence of the solver for traveltimes and k-convergence for spatial derivatives when using a P^k polynomial approximation space. The residual is defined by the value of $\mathscr{H}$ (Eq. 2) at steady state, while the error definition comes from numerical and analytical solutions (Fig. 1). In practice, we only obtain a first-order convergence of the solver for both error and residual when the standard formulation is involved due to the source singularity, whatever the approximation space we choose. However, a drastic improvement is obtained when using the additive factorization formulation (Eq. 5), such that levels of error are much lower and exhibit the expected second-order convergence of the solver. These results hold for both Cartesian and UJ discretizations (see results for P^1 approximation on Figure 1). While small differences

are visible between Cartesian and UJ cases without factorization, we clearly notice that results obtained with factorization are perfectly aligned.

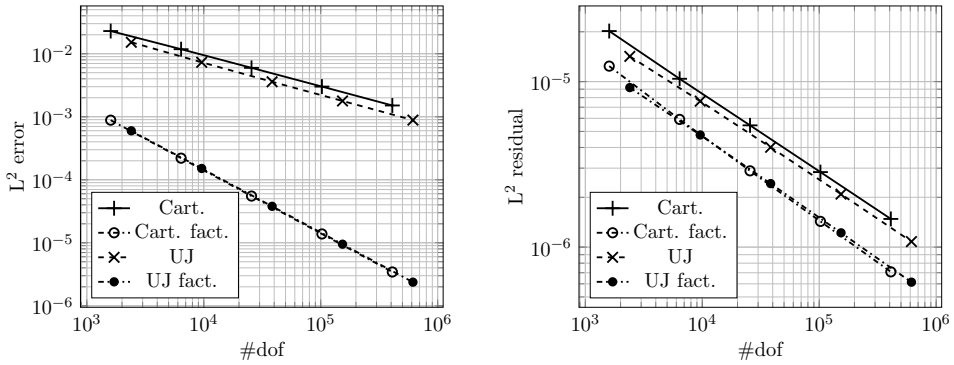

Figure 1 *Vertical velocity gradient case: errors and residuals for P^1 approximation for both Cartesian and Union-Jack discretizations with and without factorization.* #dof *indicates the number of degrees of freedom. One can see the dramatic decrease in error and residual estimations with proper convergence rates when considering factorization.*

Next we apply the DG time-marching solver on the 2D BP TTI benchmark model (Shah, 2007) with a Gaussian smoothing of characteristic lengths of 367 meters in both x and z directions in order to mitigate the impact of the discretization. The model is described by a highly contrasted P-wave velocity over a distance of 79 km and a depth of 11 km and corresponding Thomsen's parameters ε, δ and θ (Fig. 2). The DG solver proceeds over an unstructured mesh of 55 012 P^1 triangular elements. The corresponding number of degrees of freedom is 165 036. We use a finite-difference TTI solver (Waheed et al., 2015; Tavakoli F. et al., 2015) for comparison, which proceeds over a 154×1071 grid so that the number of degrees of freedom is similar (164 934). The source is located at $x = 40$ km, $z = 0$ m. Wave fronts obtained with the two methods exhibit similar shapes (Fig. 3), although small differences of less than 150 m appear at large offsets (above 20 km).

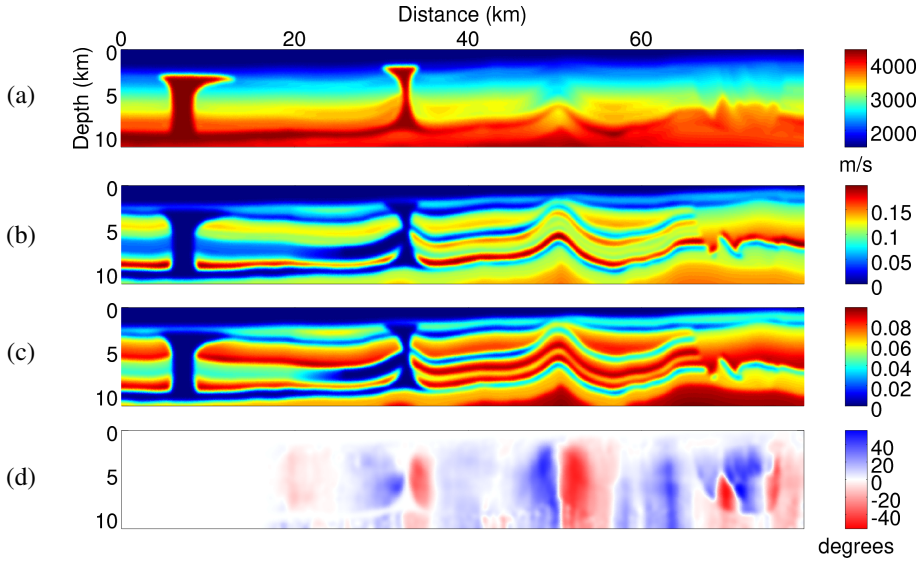

Figure 2 *The smooth BP TTI model: from top to bottom, vertical velocity (a), Thomsen's epsilon parameter (b), Thomsen's delta parameter (c), tilt angle (d).*

Conclusions

We have shown that computing directly the traveltime at fixed points is possible for complex anisotropic media using a recently designed discontinuous finite-element method. Adding outgoing conditions at the borders of the finite domain including the topography is important for Earth sciences applications. Factorization for handling the intrinsic singularity at the point source enables us to recover expected convergence behavior of finite-element schemes. Various applications show that this new approach provides high-quality solutions when compared with finite-difference methods. Complex topographies

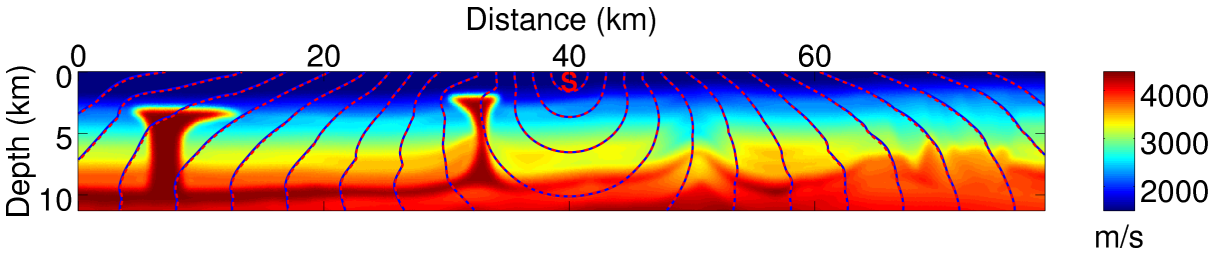

Figure 3 *First-arrival traveltime isochrones in the smooth BP TTI model computed with the DG time-marching solver (blue plain line) and with the FD fast-sweeping solver (red dashed line).*

are handled efficiently by using an unstructured mesh. Time integration is rather a slow procedure and should be combined with faster methods when one looks for the steady-state solution. Therefore we shall investigate how to incorporate this local solver inside a fast sweeping technique to take benefit of the hyperbolic structure of the equation, among other alternatives for using such solver. The quality of the solution depends on the mesh and on the interpolation order we select in elements. We shall take full benefit of the so-called hp-adaptivity in the future. We have found that this solver provides accurate solutions for isotropic and anisotropic media, opening doors to possible integration in tomography and migration tools. Finally we focus on first arrivals here, but the flexibility of the Hamiltonian formalism coupled with the extension to higher dimensions should allow for multi-arrival consideration in a phase space framework, subject to future investigations.

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