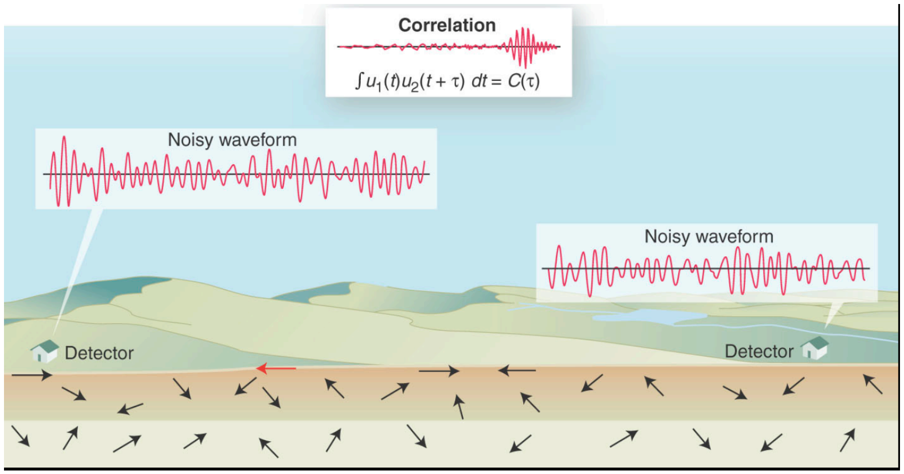
Field-field correlation
and the Retrieval of Green's Function from noise,
A Perspective from Ultrasonics

-R L Weaver
-University of Illinois



Theorem:

Noise Correlation "=" Green's Function

When is this true?
What are the needed clarifications and caveats ?

Definition of the Field-Field correlation

Noise record ψ at positions x,y

$$C(\vec{x},\vec{y};\tau) \equiv \int \psi(\vec{x},t)\psi(\vec{y},t+\tau)dt$$

"lapse time τ "

$$Or\ldots\ldots < \psi(\vec{x},t)\psi(\vec{y},t+\tau) >$$

Theorem: Should be equal (*with caveats and clarifications*) to

$$G(\vec{x},\vec{y};\tau)$$

the medium's Green function, representing the response you would have at position $\vec{x}$ given an impulse at $\vec{y}$

That is, by cross-correlating random noise, we can construct what we'd get if we could do an active experiment using artificially generated waves.

Potentially very convenient! Especially in Seismology

Plan for today:

Some Proofs of C ~ G

Two early laboratory demonstrations with ultrasound

Some practical limitations

ghost features

spurious features

signal to noise in C

Then something related, but more recent, from our lab

maybe with seismological implications

The simplest proof involves a common definition of a fully diffuse field, from room acoustics, or from the physics of thermal phonons, in terms of the normal mode expansion for the field in a finite body

$$\psi(\vec{x},t) = \mathrm{Re}\sum_{n=1}^{\infty} a_n\, u_n(\vec{x})\exp(i\omega_n t)$$

For which we assert modal amplitude statistics

$$< a_n a_m^* > = \delta_{nm} \quad 2F(\omega_n)/\omega_n^2 \qquad \text{"equipartition"(*)}$$

n.b: this follows from maximum entropy where F ~ energy per mode ($k_B T$)

It is then straightforward to derive

$$C(\tau) \equiv < \psi(\vec{x},t)\psi(\vec{y},t+\tau) > =$$

$$\mathrm{Re}\sum_{n=1}^{\infty} F(\omega_n)\, u_n(\vec{x})u_n(\vec{y})\exp(-i\omega_n\tau)/\omega_n^2$$

(*) One major assumption for the derivation

$$C(\tau) \equiv < \psi(\vec{x},t)\psi(\vec{y},t+\tau) > =$$

$$\sum_{n=1}^{\infty} F(\omega_n)u_n(\vec{x})u_n(\vec{y})\cos(\omega_n\tau) / \omega_n^2$$

Compare with the modal representation for G . . .

$$G_{xy}(\tau) = H(\tau) \sum_{n=1}^{\infty} u_n(\vec{x})u_n(\vec{y})\sin(\omega_n\tau) / \omega_n$$

(*H* = *unit step function*)

They differ by H(τ), sin *vs* cos, and F(ω)

We may conclude

$\partial C / \partial \tau = -\{G - G^{\text{time reversed}}\}$ convolved with $F(\tau)$

Support only at positive time τ

Support only at negative τ

Another proof ,

based on G's role as a *propagator of initial conditions*

$$\psi(\vec{r},t+\tau) = \int d\vec{a}\ \ \psi(\vec{s}+\vec{a},t)\ \dot{G}(\vec{s}+\vec{a},\vec{r};\tau)\ +\ \int d\vec{a}\ \ \dot{\psi}(\vec{s}+\vec{a},t)\ G(\vec{s}+\vec{a},\vec{r};\tau)$$

ψ at position *r* and a later time t + τ

may be constructed in terms of an integral of ψ

over all space at an earlier time t.

Now construct our noise correlation $C_{s\rightarrow r}(\tau)$

$$C(\tau) \equiv < \psi(\vec{r},t+\tau)\psi(\vec{s},t) > = \int d\vec{a}\ \ < \psi(\vec{s}+\vec{a},t)\psi(\vec{s},t) > \dot{G}(\vec{s}+\vec{a},\vec{r};\tau)$$

$$+ \int d\vec{a}\ \ < \dot{\psi}(\vec{s}+\vec{a},t)\psi(\vec{s},t) > G(\vec{s}+\vec{a},\vec{r};\tau)$$

(true regardless of diffuseness)

$C_{s\rightarrow r}(\tau)$ is seen to be a *spatial convolution* of the

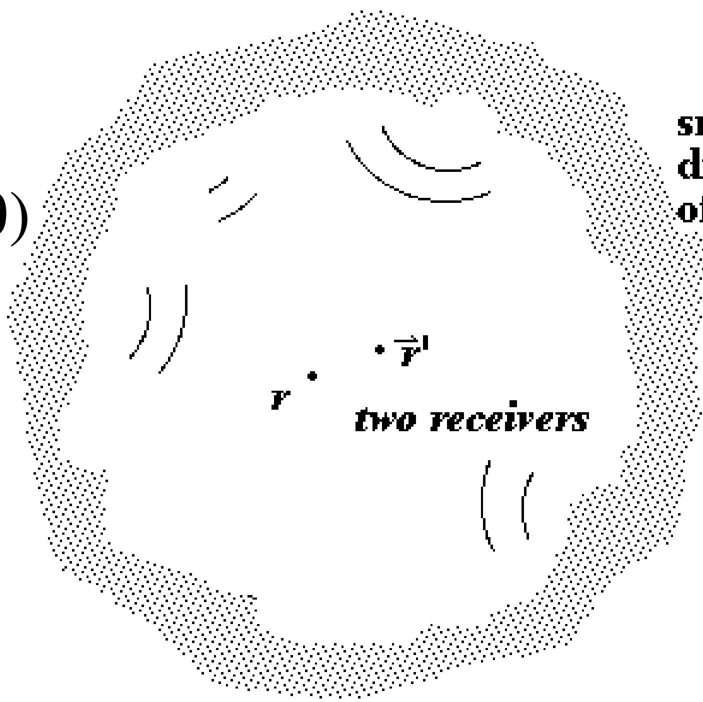
<u>equal time noise correlation</u> C(τ =0)* with G

(*) C(τ=0) is related to *Specific Intensity*, of *Radiative Transfer Theory*

A proof based on assumption that wave propagation is ballistic:

Imagine a homogeneous medium with incoherent sources at infinity

B(θ)

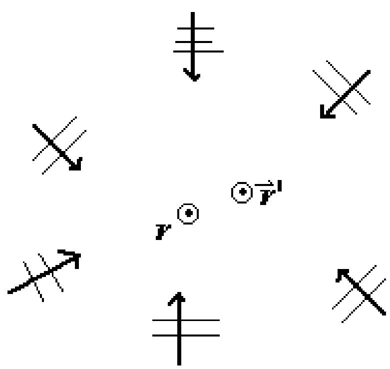


It produces a diffuse Intensity distribution B(θ) incident upon a region containing our two receivers

The field in the vicinity of the origin is a superposition of plane waves

$$\tilde{\psi}(\vec{r},\omega) = \int A(\theta)\exp(-i\omega\hat{\theta}\cdot\vec{r}\,/\,c)\;d\theta \qquad (\text{2-d})$$

with $\quad < A >= 0; \qquad < A(\theta)A^*(\theta') >= B(\theta)\delta(\theta-\theta')$



i.e, incident plane waves with intensity B(θ)

This is a ray-picture of a diffuse field, (<u>*fully*</u> *diffuse* if B = constant in θ)

This implies that the field-field correlation is

$$< \tilde{\psi}(\vec{r},\omega)\;\tilde{\psi}(\vec{r}\,',\omega)^* >= \int B(\theta)\exp(-i\omega\hat{\theta}\cdot(\vec{r}-\vec{r}\,')\,/\,c)\;d\theta$$

If special case B(θ) = constant ('*fully diffuse*')

$$< \tilde{\psi}(\vec{r},\omega)\ \tilde{\psi}(\vec{r}\,',\omega)^* >= B \int \exp(-i\omega\hat{\theta}\cdot(\vec{r}-\vec{r}\,')/c)\ d\theta$$

$$= 2\pi B\ J_0(\omega\,|\,\vec{r}-\vec{r}\,'\,|/c) \sim \operatorname{Im} G \sim G - G^{TR}$$

we recover the previous theorem.

If B (θ) $\neq$ constant,

and if $\omega\,|\,\vec{r}-\vec{r}\,'\,|/c >> 1$, we can evaluate by stationary phase*(*)*

$$< \tilde{\psi}(\vec{r},\omega)\ \tilde{\psi}(\vec{r}\,',\omega)^* >\sim B(0) \int \exp(-i\omega\,\cos\theta\,|\,\vec{r}-\vec{r}\,'\,|/c)\ d\theta$$

$$\sim\ B(0)\exp(-i\omega\,|\,\vec{r}-\vec{r}\,'\,|/c)/\sqrt{\omega\,|\,\vec{r}-\vec{r}\,'\,|/\pi c}$$

Which looks like the asymptotic form for the Hankel function, i.e., G

Thus the identification C $\sim$ G is retained in the asymptotic limit, $\omega\,|\,\Delta r\,|\,/\,c >> 1$

But…. proportionality depends on intensity B(0) in the "on-strike" direction

(*) Snieder 2004

If B $\neq$ constant, then…

$$C = \langle \psi(\vec{r},t)\psi(\vec{r}\,',t+\tau)\rangle = \int B(\theta)\exp(-i\omega\hat{\theta}\cdot(\vec{r}-\vec{r}\,')/c + i\omega\tau)\; d\theta\,\tilde{S}(\omega)d\omega$$

We evaluate in the asymptotic limit of large receiver separation

wavelet S(t) related to power spectrum of noise

$$= \frac{-1}{4\pi}\sqrt{\frac{2\pi}{\omega x}}\int_0^{+\infty} d\omega\, i \exp(i\omega(\tau - x/c))\tilde{S}(\omega) \quad \times$$

$$\{B(0)e^{i\pi/4} + B''(0)\tfrac{1}{2\omega x}e^{3i\pi/4} - B(0)\tfrac{i}{8\omega x}e^{5i\pi/4}..) + c.c.$$

Leading term

first correction

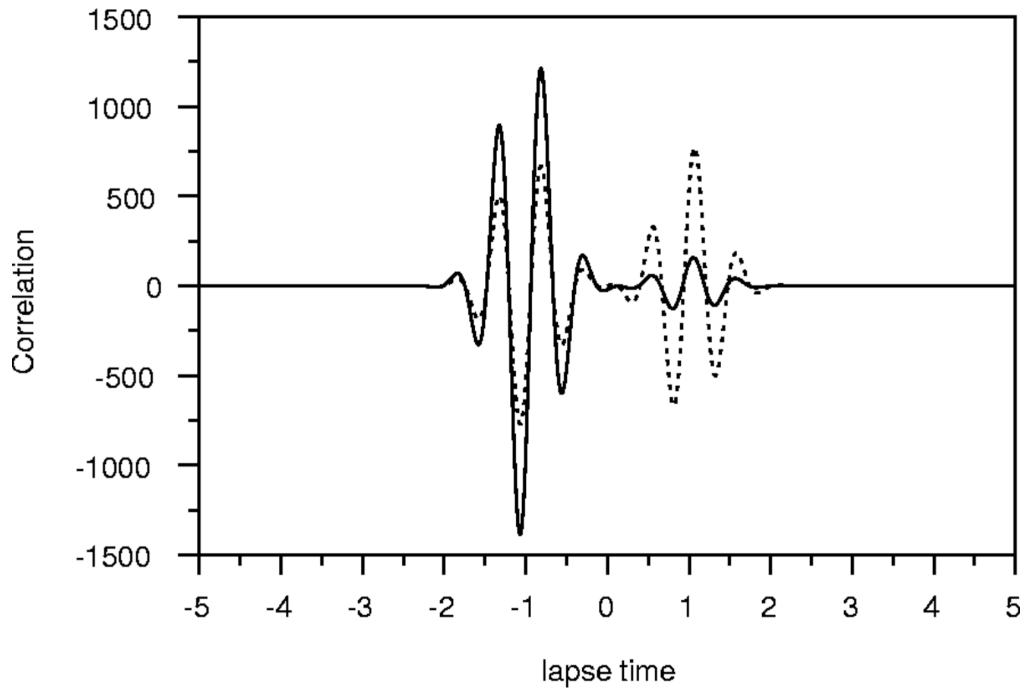
We see that the apparent arrival time is delayed relative to $|r-r'|/c$ by a fractional amount $[B''(0)/B(0)] / 2k^2|r-r'|^2$

→ The effect of non-isotropic B on arrival time is small in practice

→ Hence the high quality of typical maps of seismic velocity

In-*spite of* ambient seismic noise being not equipartitioned!

Comparison of Correlation waveform (solid line)
and time-symmetrized G (dashed line)
For case of non-trivial noise directionality $B(\theta) = 1 - 0.8 \cos \theta$



Numerical simulations

Froment et al 2009

Our rough identification is retained: C shows propagation

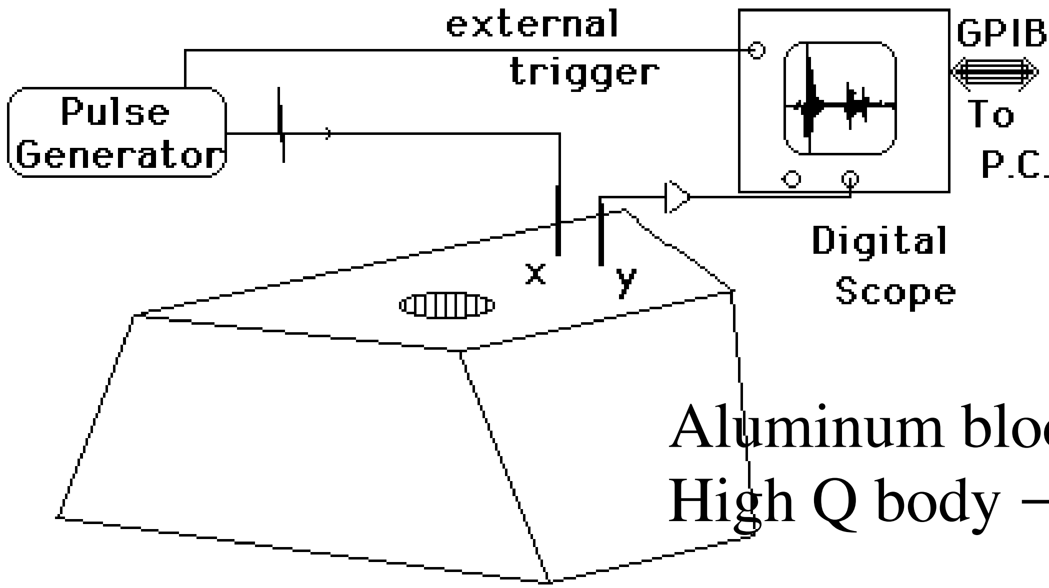
But a) precise assertion fails, $G \neq dC/d\tau$

b) large differences in amplitudes at positive and negative time

c) there are *tiny* shifts of apparent arrival time, as predicted

Laboratory Verification ?

An ultrasonic "pitch-catch" measurement

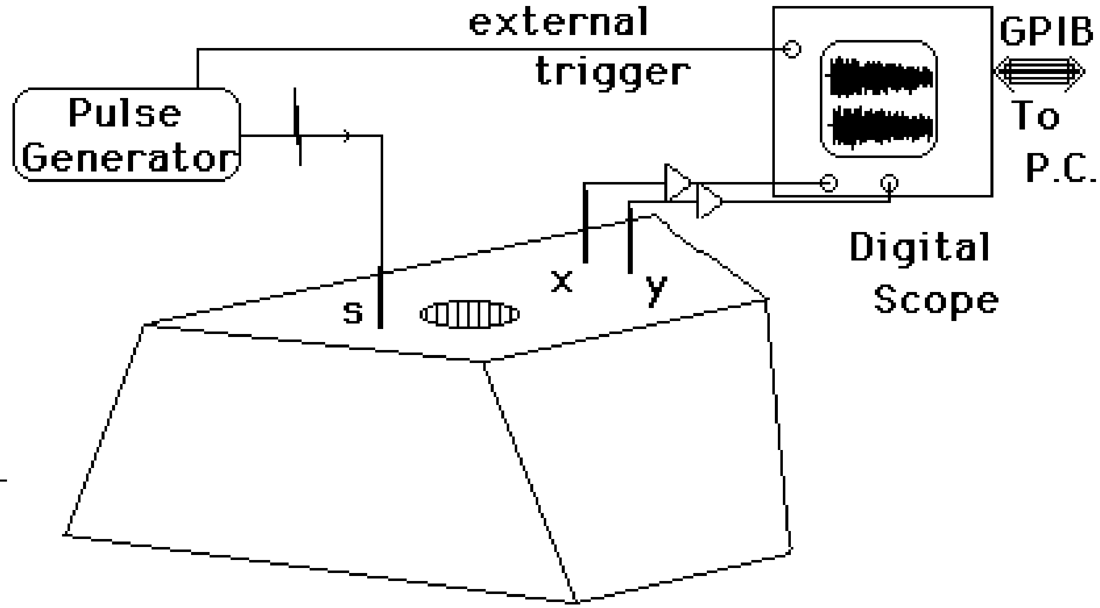


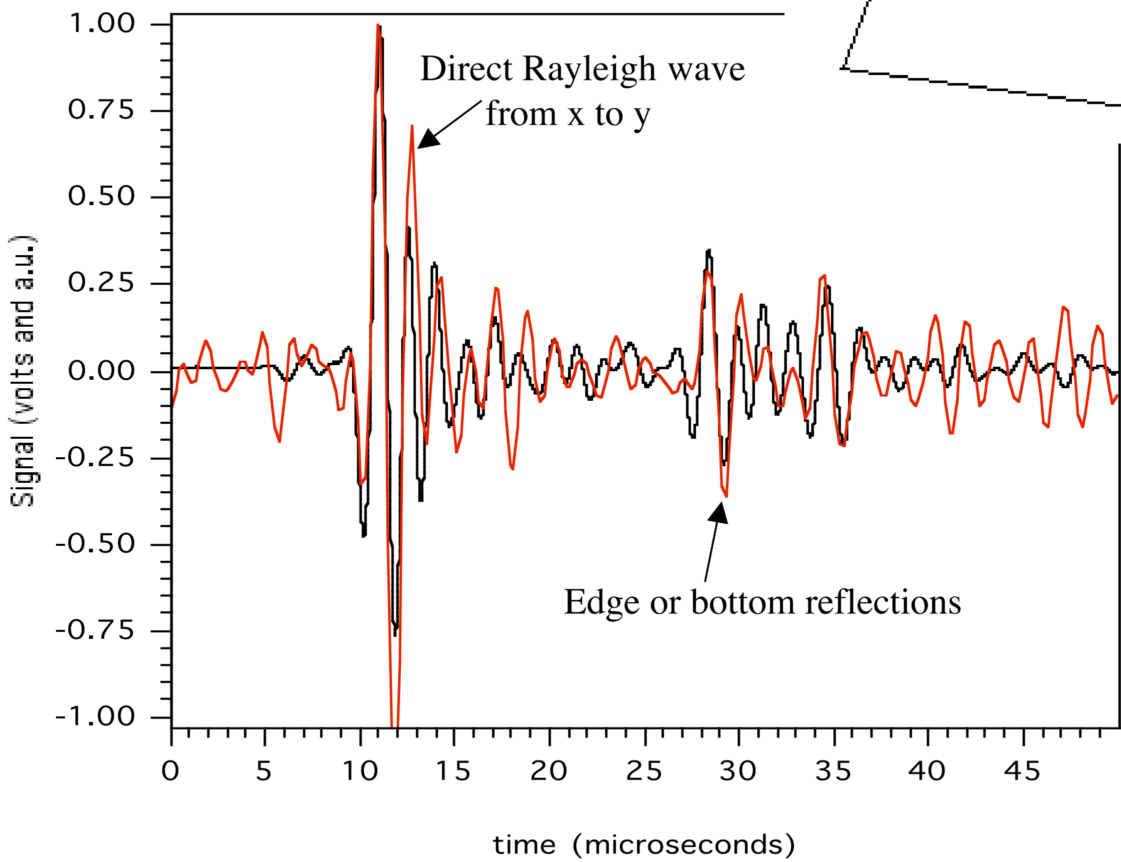
Aluminum block ~ 10 cm wide
High Q body →
Highly diffuse field

An impulse (with frequencies up to MHz) is applied at position x.

The resulting mechanical motion
(wavelengths λ ~mm, duration ~100msec)
is detected at position y.

Now apply the source at position "s" and detect the resulting motion at "x" and "y."

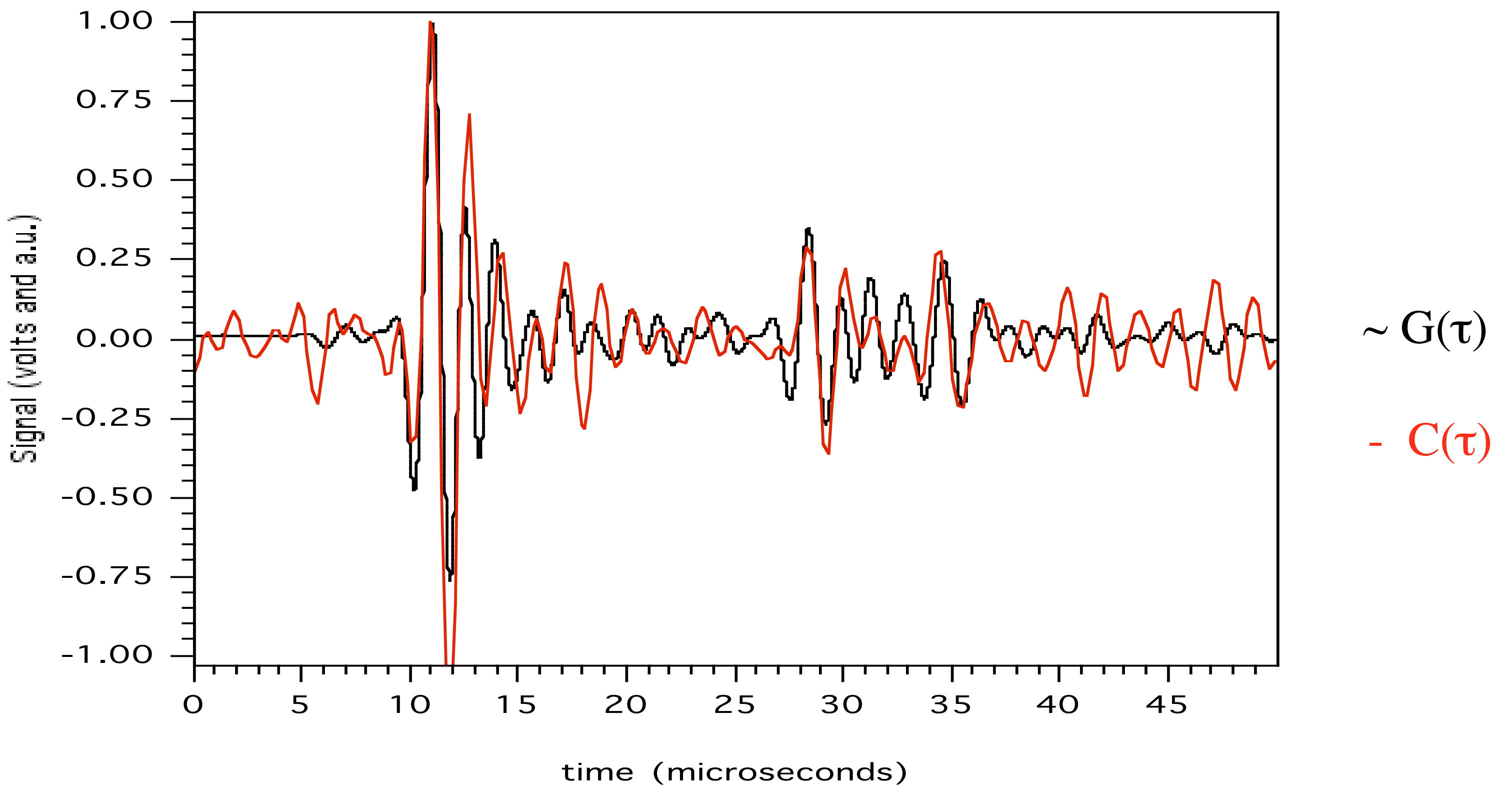




direct pitch/catch signal
x ➔ y

Correlation between signals
s ➔ x & s ➔ y

J Acoust Soc Am. **110**, (2001)



Why was correspondence imperfect?

Fields not 'fully diffuse' ?

Not enough averaging in time ?

F(ω) ?

the chief culprit

Recall

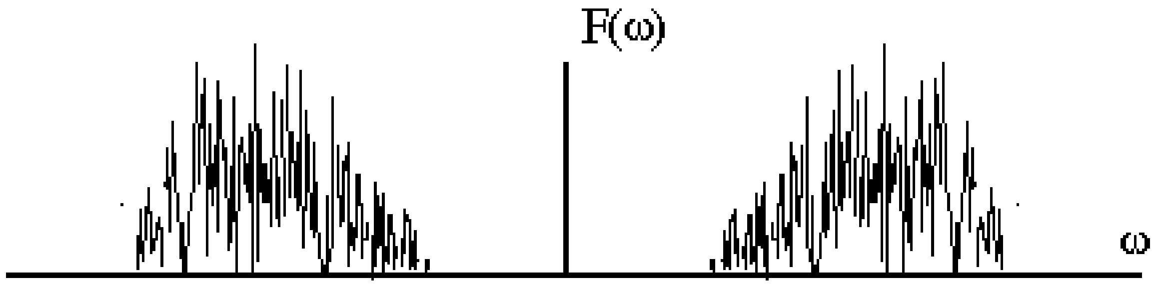
$$\partial C / \partial \tau = -\{G - G^{\textit{time reversed}}\} \textit{ convolved } \text{with } F(\tau)$$

What affects F, the spectrum of the noise ?

1) Our signal processing and filters
(not so critical, as these are compact in time)

2) The environment of the source -
e.g. reflections (especially nearby reflectors)

Typical F(ω) for a single source after filtering to a band of interest



Notice:

Smooth Envelope (due to filtering in the signal processing)

Slow undulations - related to early echoes in $G(s \rightarrow s)$

corresponding to modest time scales $O\ (1/\Delta\omega)$

Fine scale hash - related to late echoes in $G(s \rightarrow s)$

corresponding to long time scales $O\ (1/\Delta\omega)$

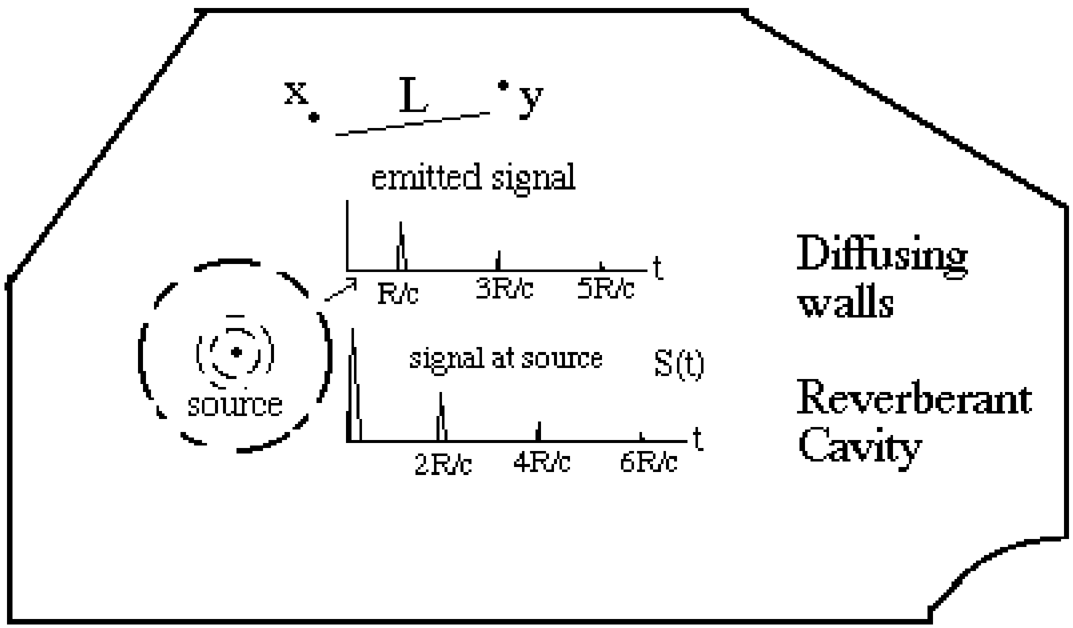
F retains information on the environment of the source s,

in particular time scales associated with backscatter $s \rightarrow s$

Therefore G convolved with F….

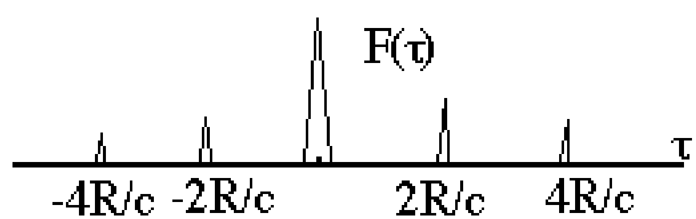
Illustration of the issue

Consider a source in the center of a semi-permeable spherical cavity Inside a chaotic larger cavity

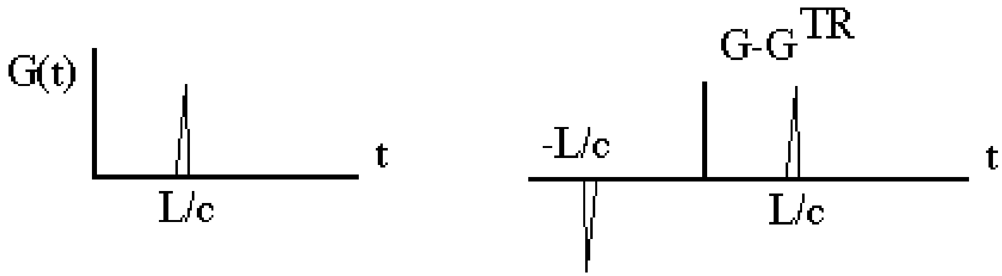


Multiple wall reflections generate a diffuse field, with statistically Isotropic intensity

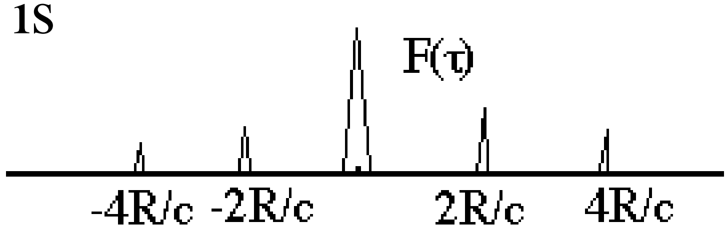
So $F(\tau) \sim S(\tau)$ convolved with $S(-\tau)$, looks like



But G and G-G^{TR} look like



while F is



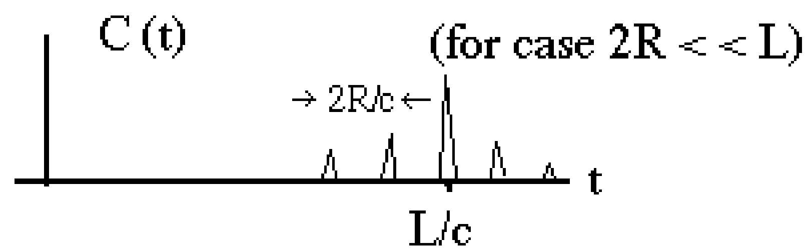
So (G-G^{TR}) convolved with F has apparent arrivals at

at $t = \pm L/c$ (these are the ones we like)

AND at $t = \pm L/c \pm n(2R/c)$ for all $n = 1,2\ldots+\infty$

I call these "ghost arrivals"

The Correlation then might look like this………



Lesson -

Even in a very diffuse field (well mixed, with lots of scattering),

you can retrieve a bad empirical Green function

due to $F(\omega)$ having fine structure.

How to fix this issue?

make sure your $F(\tau)$ is compact in time

i.e that your noise spectrum is smooth in frequency

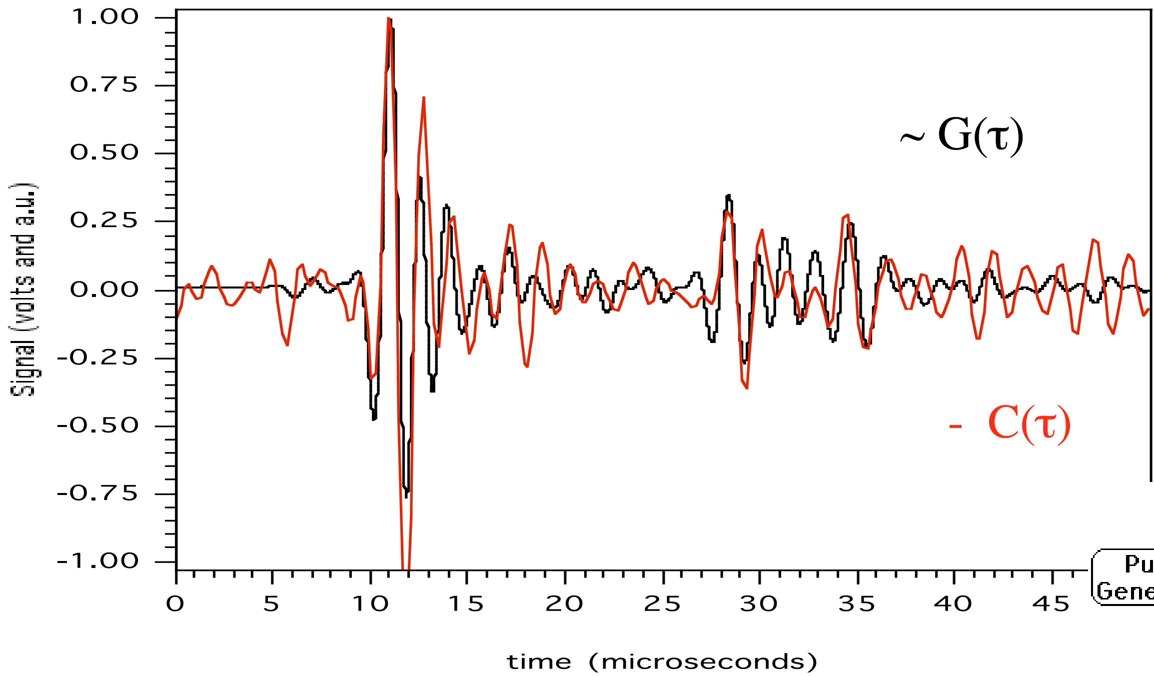
this is a property of the source

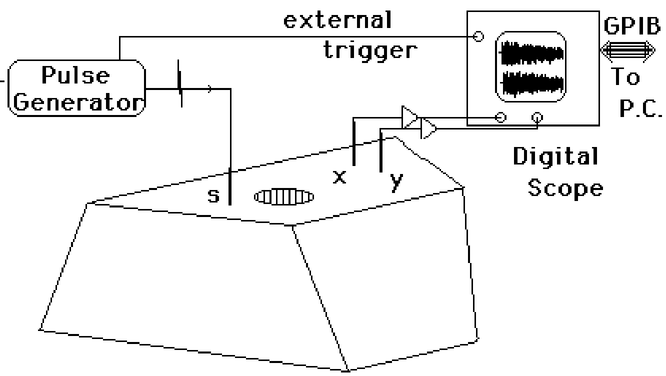
Standard fix: Sum over many sources,

at a multitude of positions,

preferably each far from prompt reflections

and such that reflections tend to cancel





Why was correspondence imperfect?

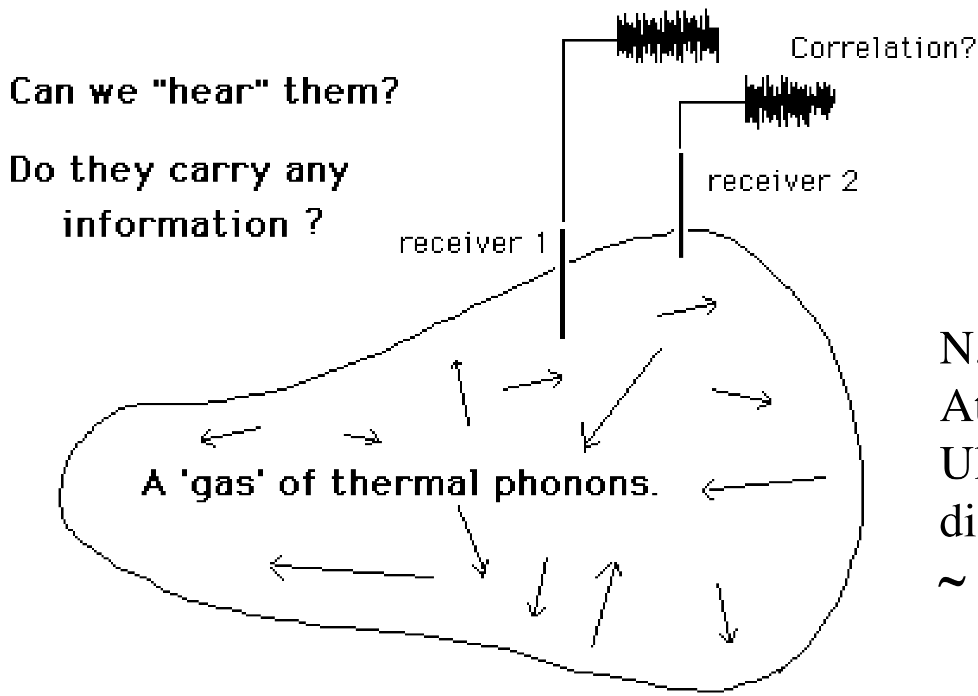
F(ω)

the chief culprit.
Because I only used 7 source positions 's'
And all were subject to wall reflections.
So Average F(ω) was not smooth

Another way to fix it is to use thermal fluctuations of elastic waves

A gas of phonons as it were . . .

with guaranteed smooth F(ω)



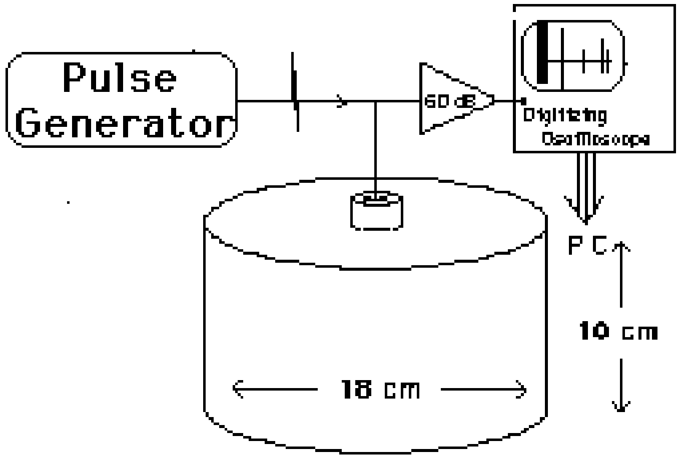
N.B.
At room temperature
Ultrasonic thermal displacements are
~ 10^{-15} m / $\mathrm{MHz}^{1/2}$

-> very weak

Laboratory verification in the thermal case:

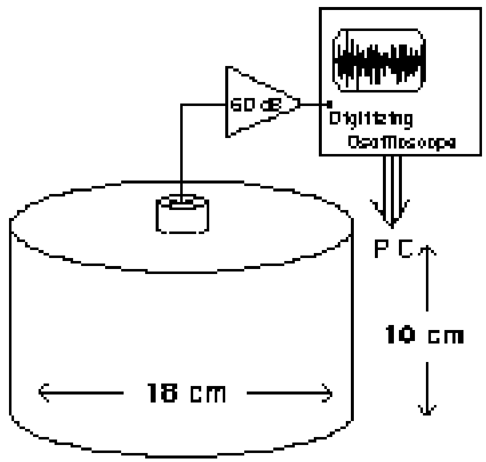
Comparison of a Direct Pulse-Echo Signal,

(conventional ultrasonics)



and

Thermal Noise Correlation

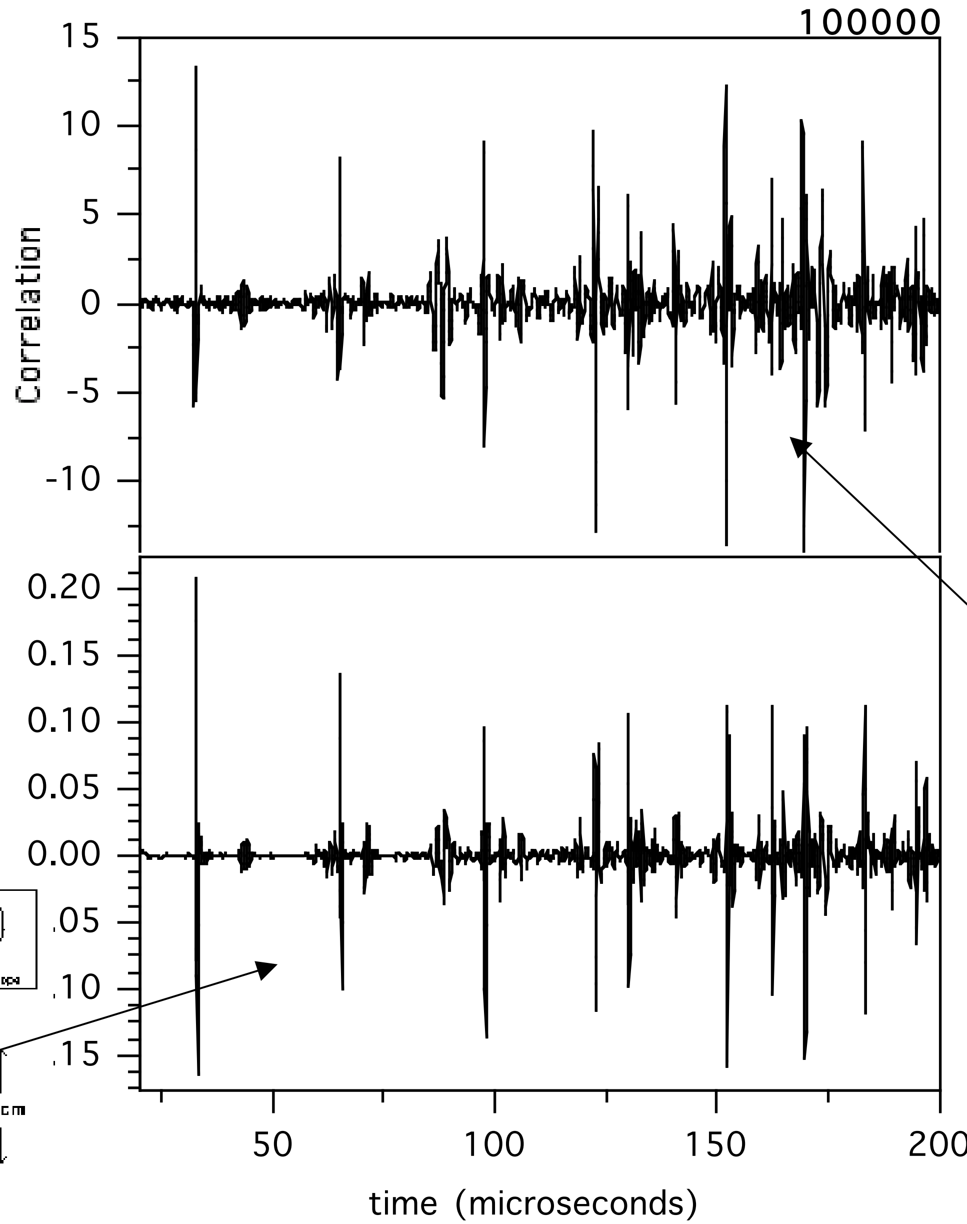


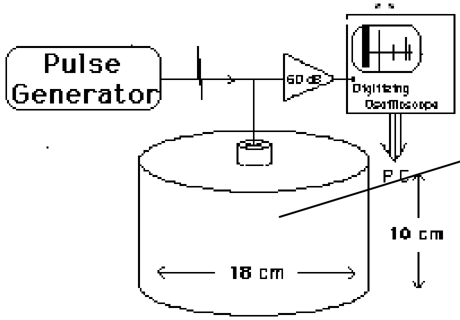
Correlation
Of
Thermal noise

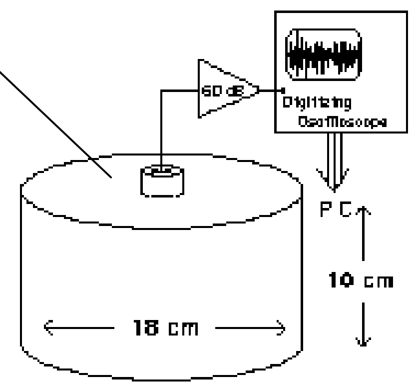
After
Capturing
320 seconds
Of data

(and taking 2.5
hours to do so)

Direct
Pulse-Echo
Signal



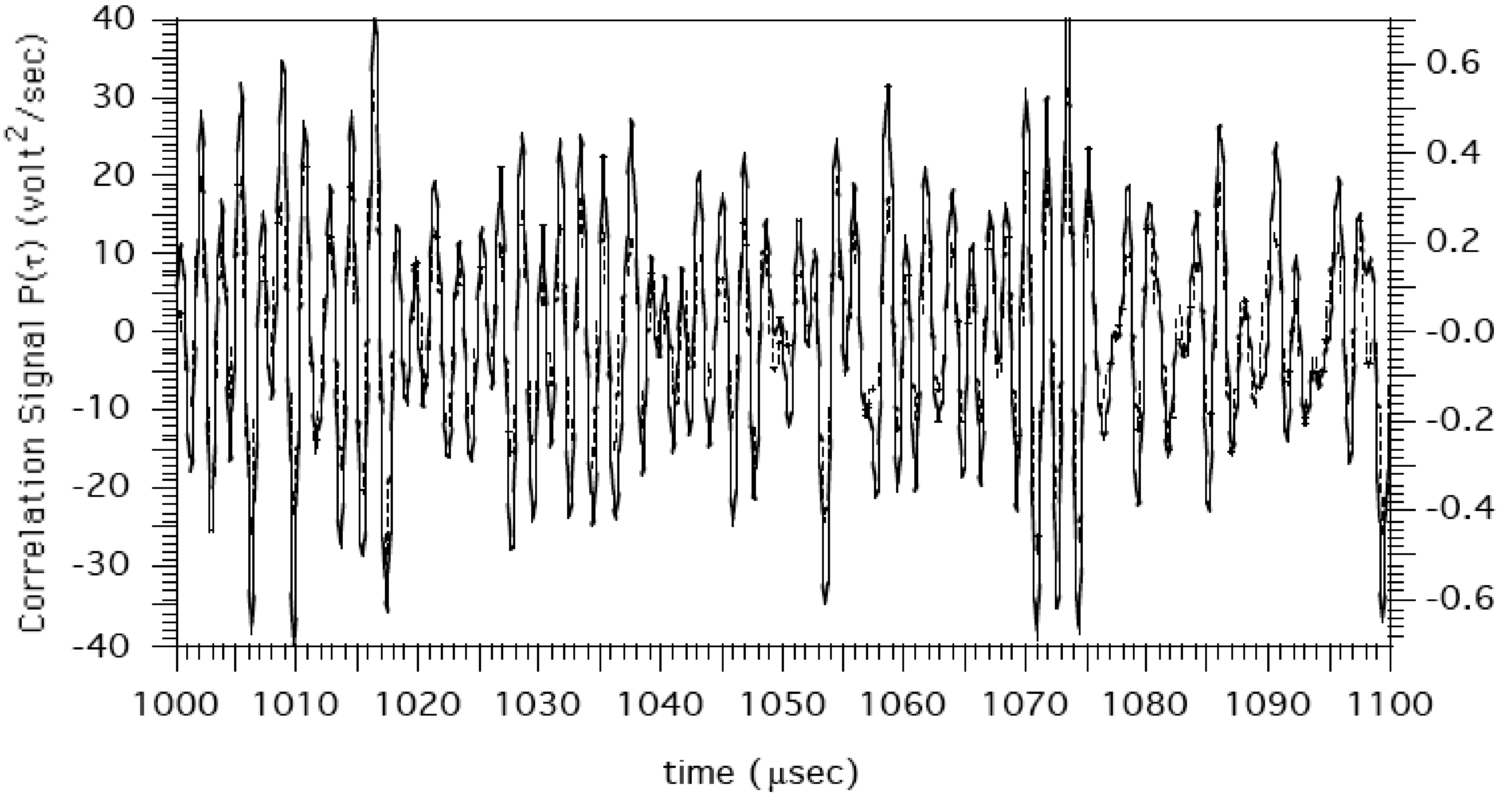




Phys Rev Lett **87** 134301 (2001)

Comparison at later times

(~ 1 msec, after rays have traveled ~3 meters)



So in the <u>ideal</u> case of a

- ⇨ Fully diffuse, equipartitioned, noise field
- ⇨ And a smooth spectrum F(ω)

We recover G very well.

In practice, one or both of these conditions may not be met

In seismology at ~10 seconds,

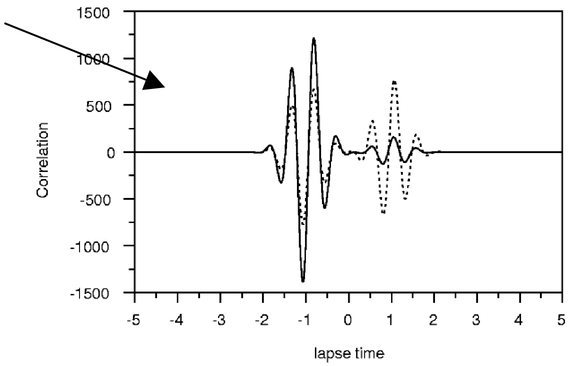
Full equipartition (full diffuseness) is rarely met.

Spectrum smoothness is ok?

What consequences can ensue from imperfect equipartition?

Consequences of imperfectly partitioned noise:

➔ non-symmetric Correlations $C(\tau) \neq C(-\tau)$
(but arrival times are often still robust)

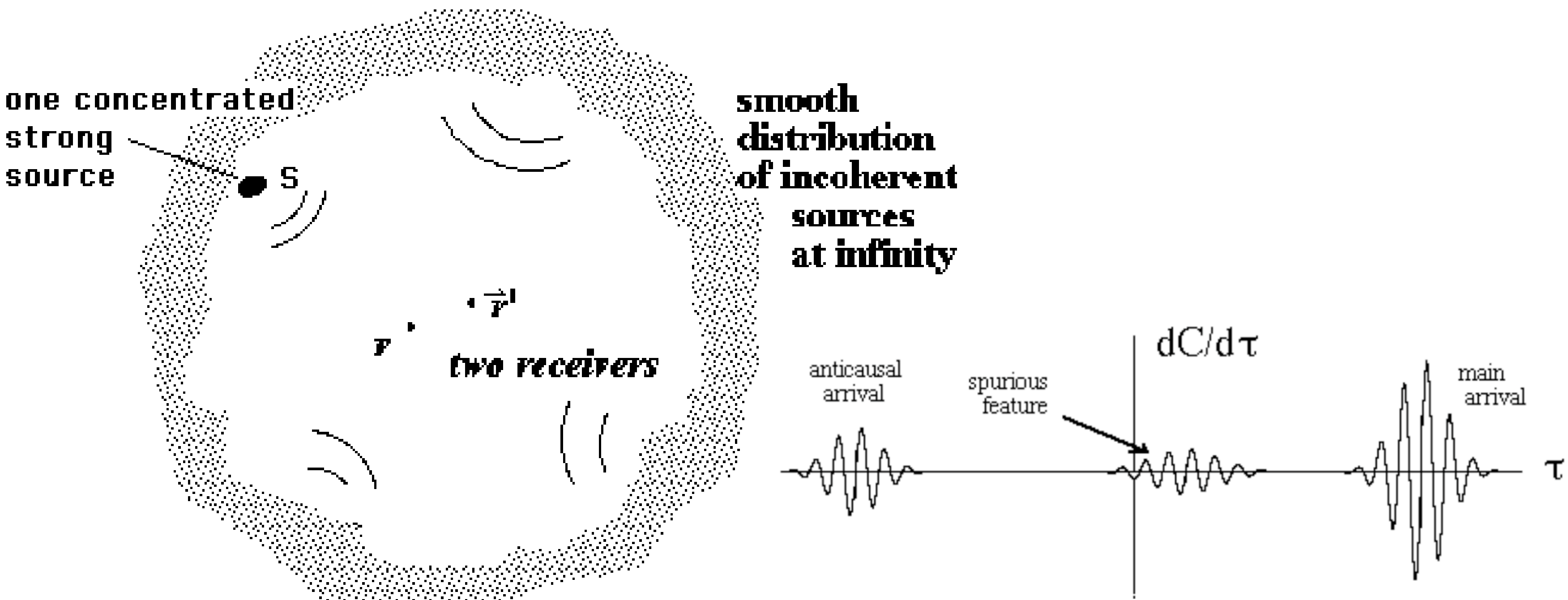


➔ Amplitude information is hard to
interpret … because it depends on
noise intensity B(θ) in the on-strike direction

➔ Spurious features in the correlations

One consequence of non-fully-diffuse noise occurs

if there are <u>point</u> sources of small angular size $\delta\theta < 1/k|r-r'|$



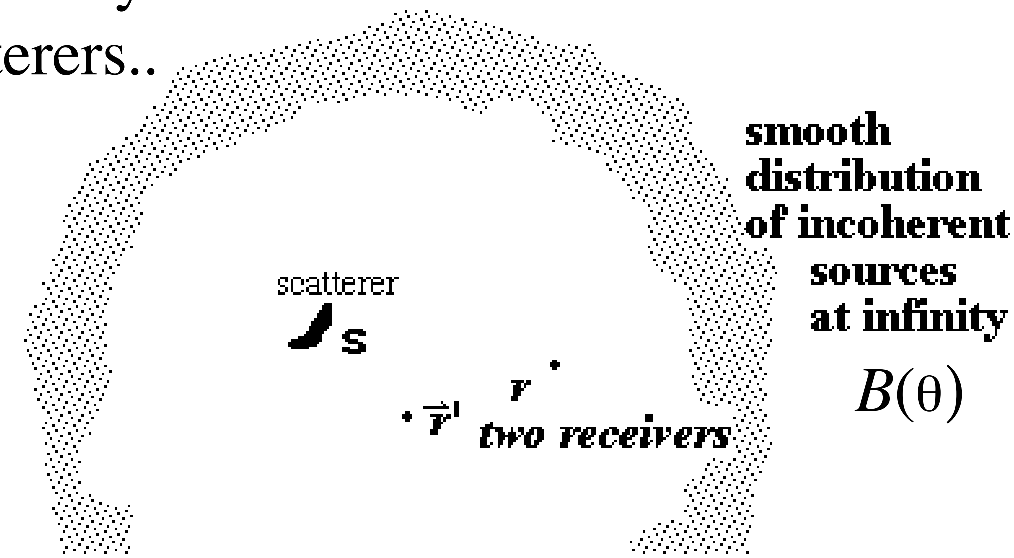
C(τ) will include a spurious arrival at a wrong time

at $\tau = |\mathbf{r}-\mathbf{S}|/c - |\mathbf{r}'-\mathbf{S}|/c < |\mathbf{r}-\mathbf{r}'|/c$

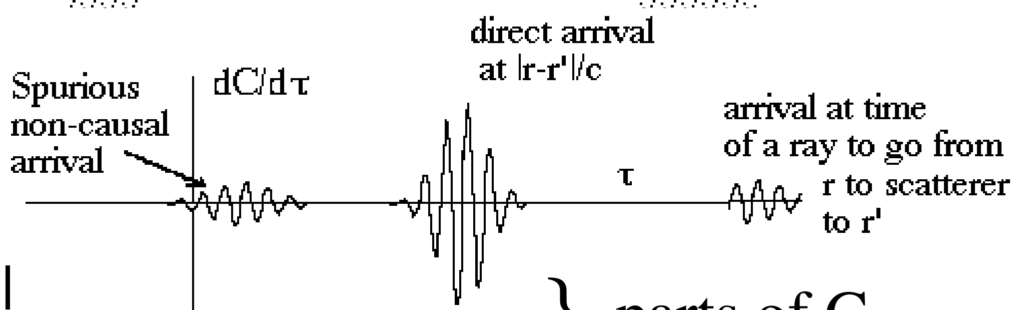
non-causal

Another consequence of non-fully-diffuse noise is spurious arrivals due to scatterers..

Intensity distribution
$B(\theta) \neq$ constant
but no point sources



Correlations in the presence of a scatterer can show



a direct arrival at $\tau = |\mathbf{r}-\mathbf{r}'|$

an indirect arrival at $\tau = |\mathbf{r}-\mathbf{s}|/c + |\mathbf{r}'-\mathbf{s}|/c$

} parts of G

and

a spurious arrival at $\tau = |\mathbf{r}-\mathbf{s}|/c - |\mathbf{r}'-\mathbf{s}|/c$

non causal

Disappears if field is Equipartitioned !!
(and scatterer is non-absorbing)

Another concern about C ~ G :

SNR (signal to noise ratio) - how much averaging is needed ?

In practice C is constructed by

$$C(\vec{x},\vec{y};\tau) \equiv \int_T \psi(\vec{x},t)\psi(\vec{y},t+\tau)\,dt$$

How much time T is required to get convergence?

SNR estimates (assuming perfectly diffuse field, and 2-d)

$$SNR = (numerical\ prefactor)\ \sqrt{Bandwidth \times T}\ \sqrt{c/\omega L}\ \exp(-\alpha L)$$

Improves with longer integration times T
and closer receiver separations L

Summary (re G function retrieval)

$C \sim G$

But be careful:

Need a (fully?) diffuse noise field

- if not fully diffuse, be aware of potential for
  - spurious arrival features from scatterers or point sources
  - amplitude distortions

Need a smooth spectrum $F(\omega)$

- non smooth F generates "ghosts"
- averaging over multiple sources can smooth F

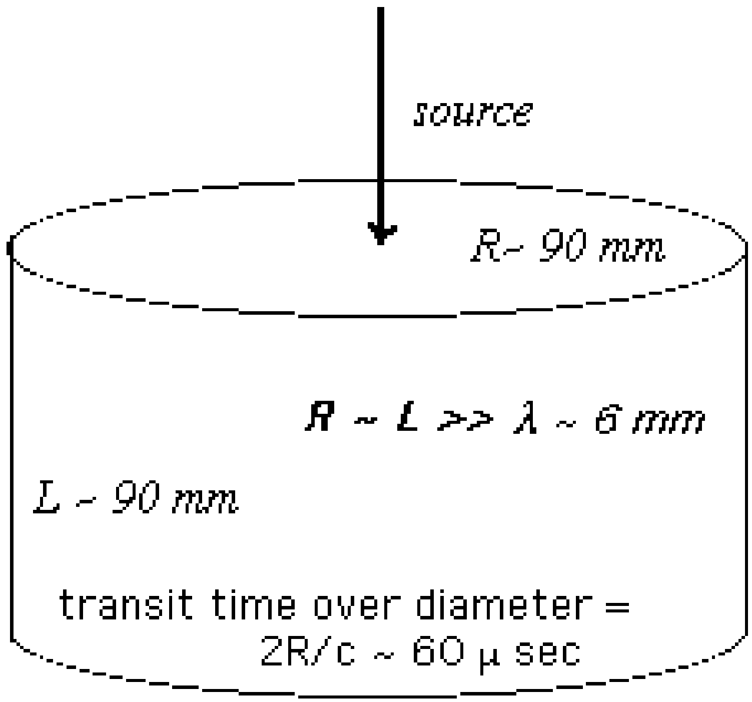
Need enough integration time T to get good SNR

Diffuse Elastic Waves
in a nearly axisymmetric body(*)

(*) like the earth??

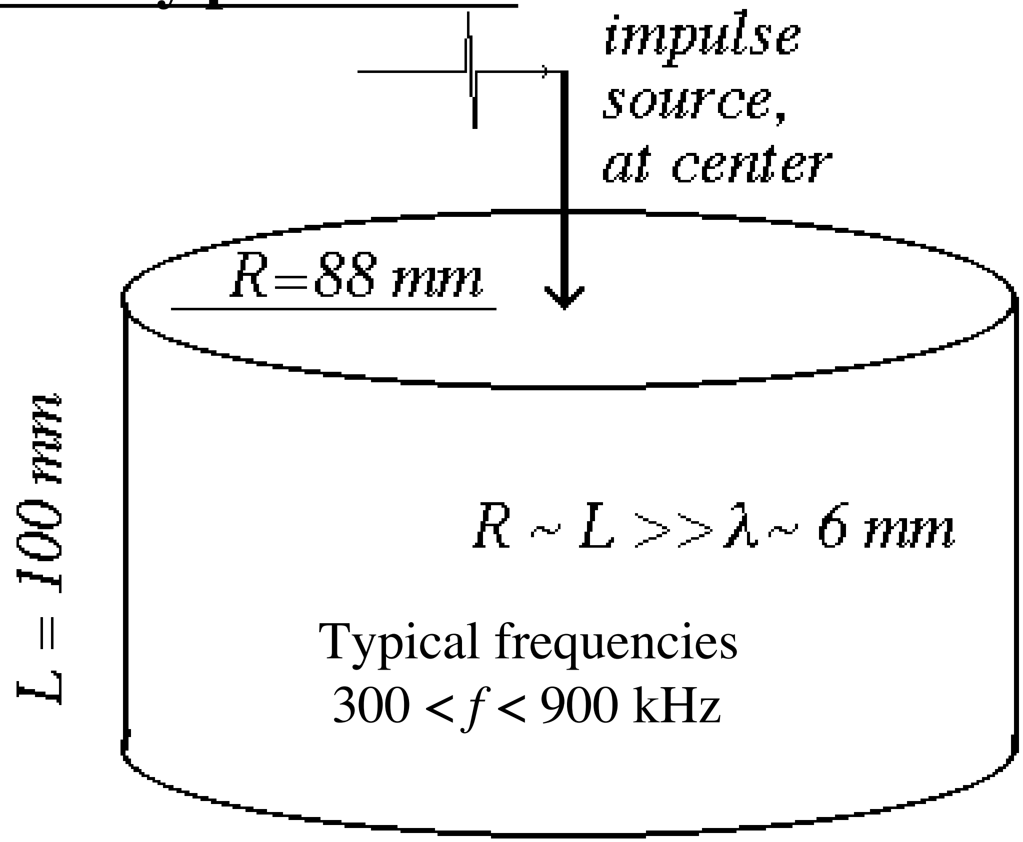
Eur Phys J 2017

We do Acoustics
with reverberant ultrasonic
elastic waves
in finite bodies



Sketch of our system.
An aluminum cylinder
With a source at the center of the top face

A few key parameters



Specimen
is large compared
to a wavelength

Specimen has a high "Q"
$\sim 10^5$

Transit time

through thickness = L/c ~ 30 μsec

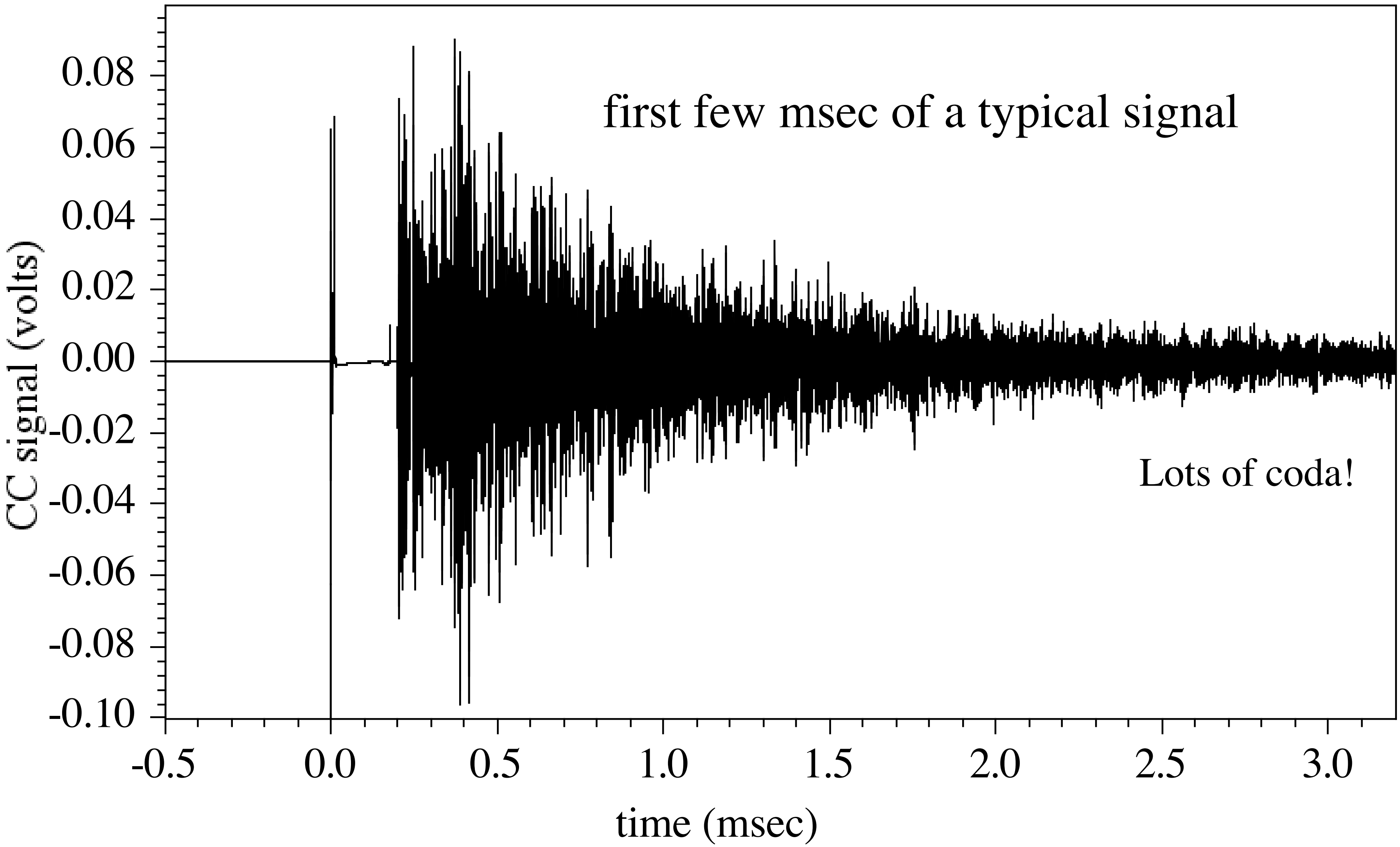
across diameter = 2R/c ~ 60 μsec

Observing time

~ 100 msec

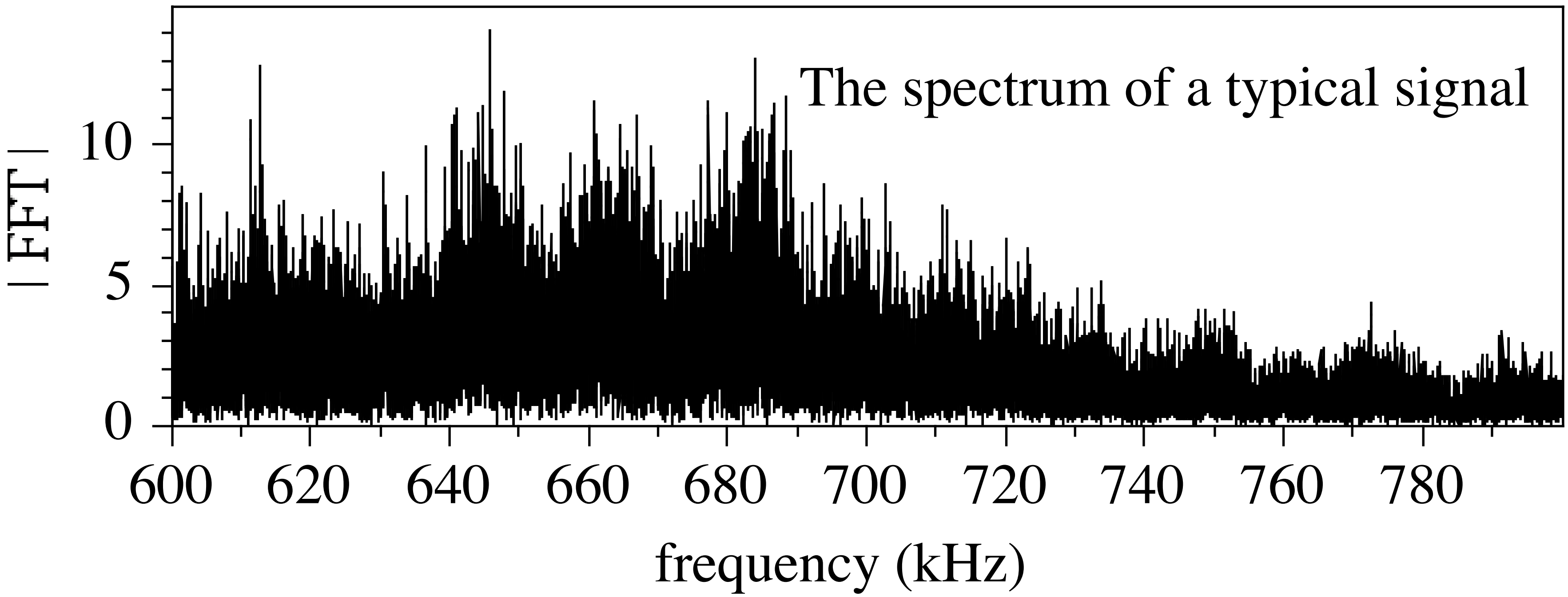
= ~2000 transits

Object is highly
reverberant



Goal of Theory: Predict amplitude of such diffuse signals,
in particular: dependence on time and position

Aside:



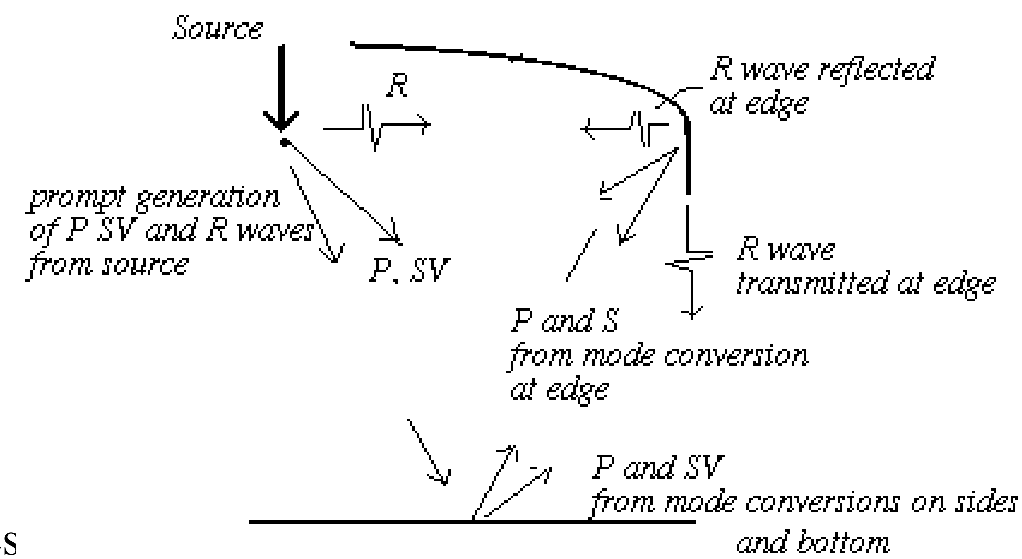
Note the undulations in the spectrum
~ frequency scales of O(20 kHz)
correspond to times scales of O(50 μsec)
= Transit times!

Note the fine scale Hash -
Corresponds to later reverberations

What do we expect this reverberant coda to consist of ?

Diffuse Field theory (as in e.g. room acoustics)
posits that each normal mode (in any narrow band) gets ~same amount of energy
I.e, after enough scattering, the energy is uniformly distributed

But that cannot be the case
if the body has special symmetries:



A source in the center of the top face of this cylinder will excite the axisymmetric m=0 modes, and no others.

Such a source will generate
$m = 0$ P and SV and Rayleigh waves
(~60% of energy into Rayleigh)

These will mode convert at the edges and surfaces
into each other… *maintaining their m=0 character*….
but redistributing amongst P and SV and R. After many reflections, most energy
will be in SV; very little in R. This takes a time of the order of a few times R/c.
i.e. less than a few 100 μ sec

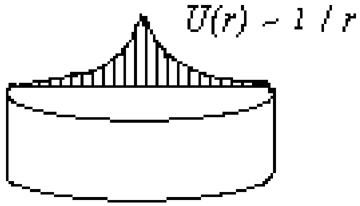
An m = 0 diffuse wave field (P/SV/R) will distribute its vertical displacement across the top surface like $J_o(kr)$ (k being a typical wave number ω/c)

Hence the measured mean square signal strength on the top surface ought vary like $U = J_o(kr)^2 \sim [2/\pi kr] \cos^2(kr-\pi/4) \sim 1/\pi kr$

That is, U(r) ought diminish rapidly with distance from the center.

In particular, the ratio of $U(0)$ = signal^2 at center
to $U(r)$ = signal^2 at distance r ought be

$$U(0)/U(r) = J_o^2(0)/J_o^2(kr) \sim \pi kr$$



For r = 50 mm, f = 500 kHz, and taking $k = \omega/(1.29c_s)$, this is about 125 (!)

Prediction : A vertical source in the center generates a diffuse field with mean square signal ~125 times greater* at the center than at half way out.

(and then of course decays in time while maintaining the distribution in r)

* And by an additional factor of 2 due to *enhanced backscatter, thus we predict a ratio of 250*

But is that the whole story?

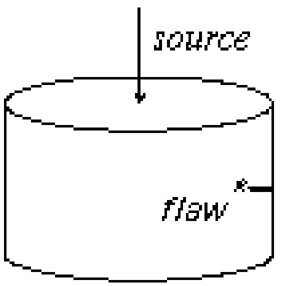
A secondary consideration:

Weak non-axisymmetry* in the body will cause scattering from $m = 0$ to other m.

* Flaws ? Supports? Transducers ? Crystallites ? Imperfect shape?

Eventually energy ought be equipartitioned amongst all states of different m….
The body will be fully equipartitioned. Energy should be uniformly distributed across the top.

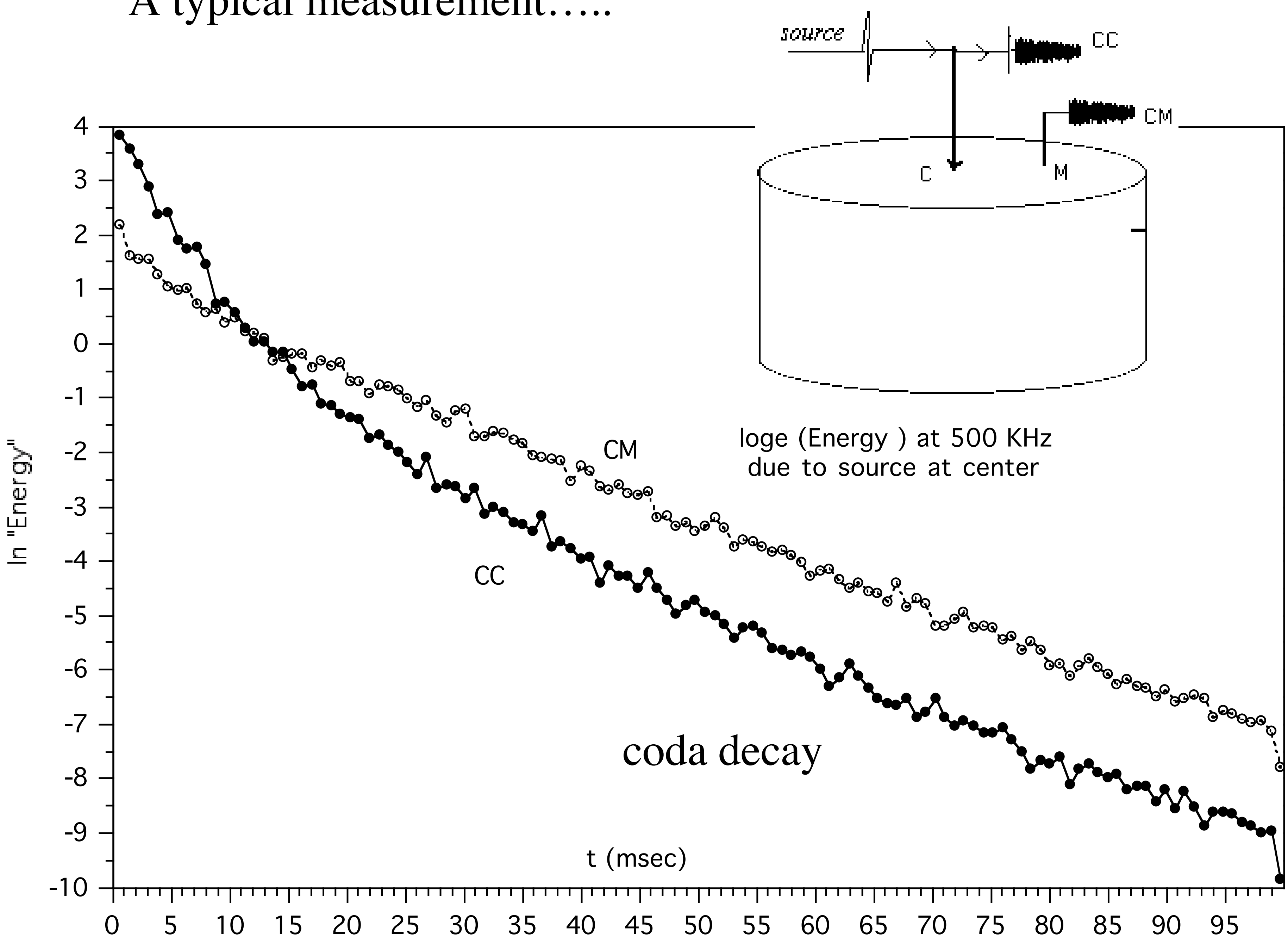
Prediction: The ratio *U(0)/U(r)*, that was ~250, should relax to 2.
(if we wait long enough) **i.e, relax by a factor of 125**
(!)



The time scale for that change
will depend on the degree of non-axisymmetry.
Different frequencies may be more or less sensitive to symmetry breaking features.

laboratory concern: Which happens faster, dissipation or transport?
i.e. will we see the transition from 250 to 2 before signal becomes inaccessible?

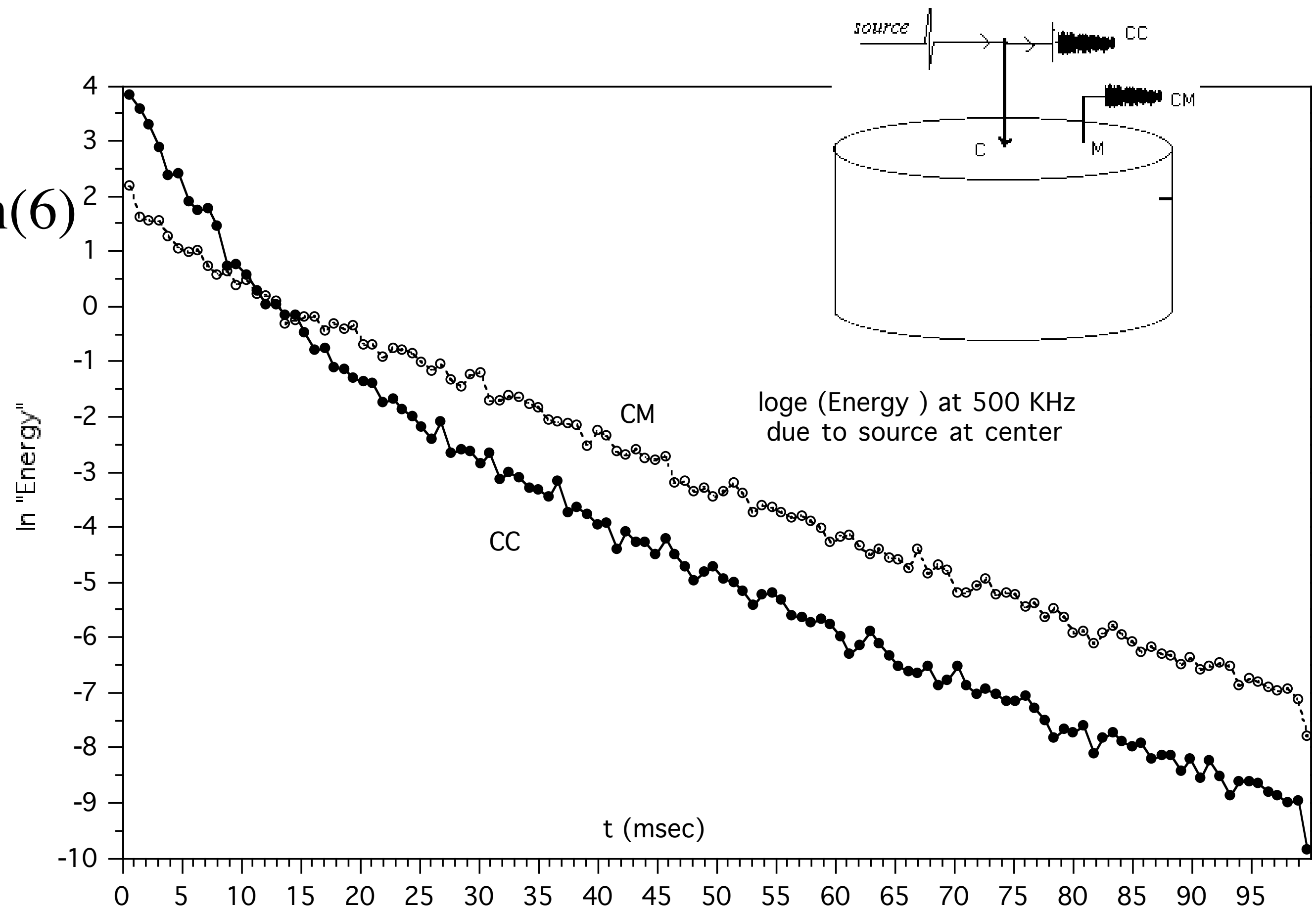
A typical measurement…..



Observe:
CC-CM at
time zero is ~1.8 = ln(6)

NOT ~ ln(250) !
Data disagree with
prediction 1.

Problem:
Sensors at
C and M are
not equally
sensitive.



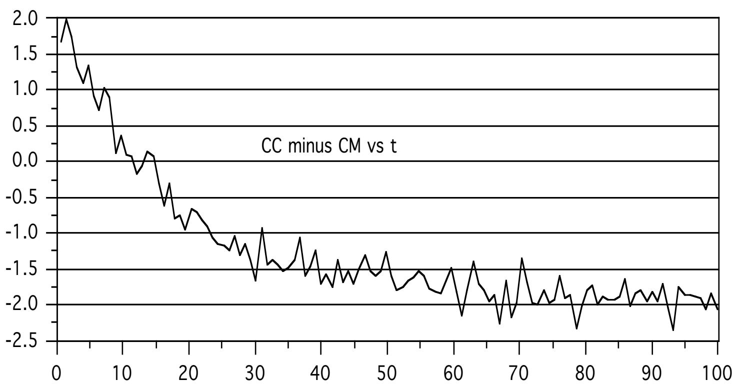
Fix(?) Calibrate by assuming (prediction 2) that at *late time* the ultrasonic energy densities at C and M are equal (i.e equipartition is achieved as $t \rightarrow \infty$)

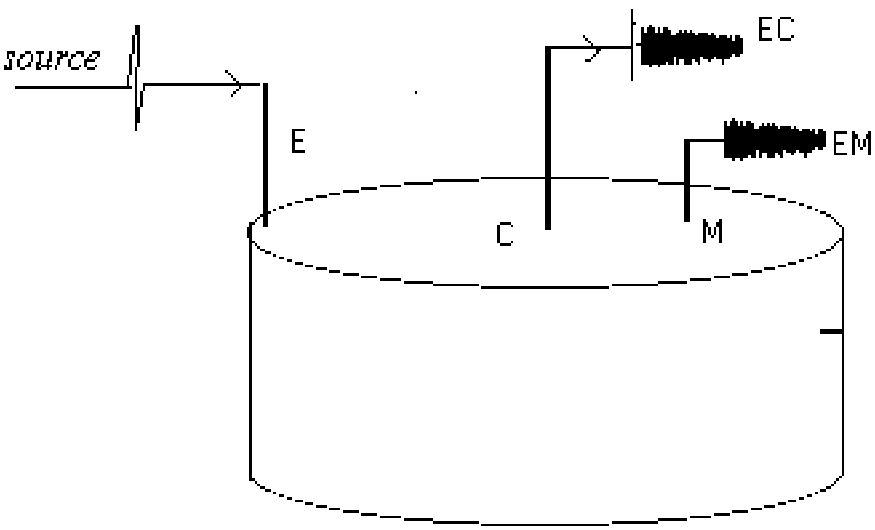
Thus ask for $(CC\text{-}CM)_{t=0}$ $-(CC\text{-}CM)_{t=\infty}$ = 4.0 = ln (55)
Accord is better but data still disagrees with prediction 1 ln(125).

Why do they disagree still?

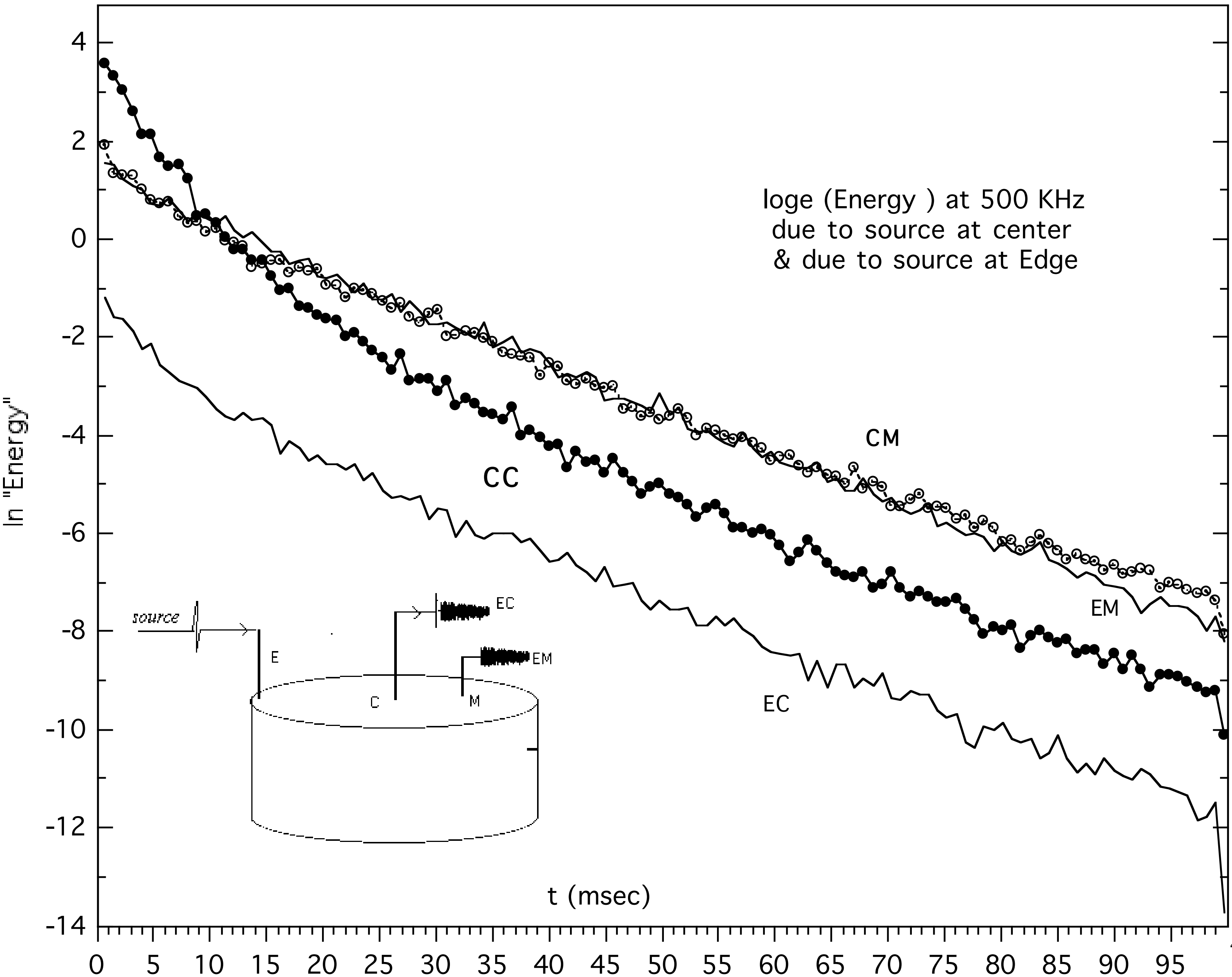
Maybe CC and CM have not achieved equipartition yet?

(parallel profiles at late time suggest however that CC-CM
will not change much at later times; (the late time steady state seems to
have been achieved)





Fix: Apply another source, at E,
And compare EC and EM
in order to calibrate C and M
(theory says E will deposit its
initial energy more uniformly
in *m* than does C, thus requiring less scattering
to achieve full equilibration)



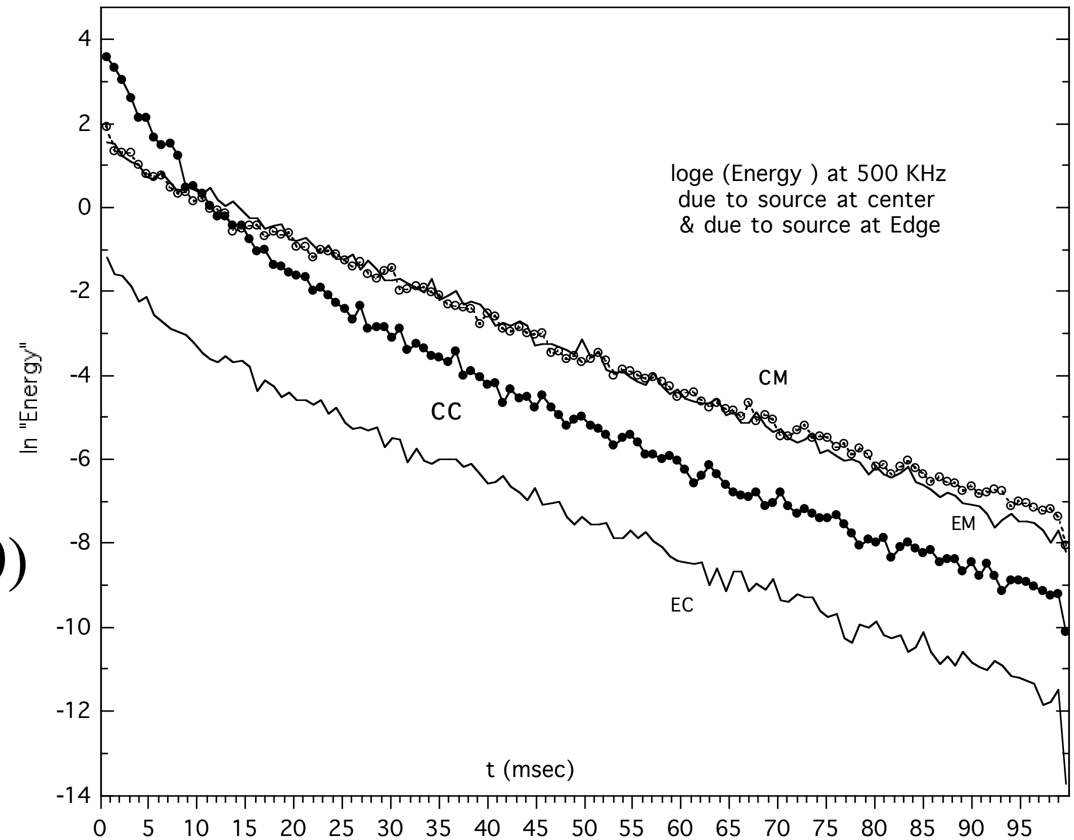
loge (Energy) at 500 KHz
due to source at center
& due to source at Edge
ln "Energy"
t (msec)
CM
CC
EM
EC
source
E
C
M
EC
EM
4
2
0
-2
-4
-6
-8
-10
-12
-14
0
5
10
15
20
25
30
35
40
45
50
55
60
65
70
75
80
85
90
95

$(CC\text{-}CM)_{t=0}$

$-(EC\text{-}EM)_{t=\infty}$

$= 5.7 = \ln 300$

(pretty good agreement with expected value 250)



Upshot: By choosing to calibrate *this* way, we get agreement.

But why didn't CC-CM go to the expected asymptote?

(put differently why isn't CC-CM @t=∞ a good calibration?)

(or, why does the energy distribution not go to equipartition?)

Do we understand what is happening here?

Do we understand what is happening here?

Answer: Yes.

We have *Dynamical Anderson Localization*.
Some energy is stuck in m=0 and cannot diffuse out to other m, even after lots of time. Scattering is too weak.

=> Equipartition is not achieved, even at $t=\infty$

This is expected to happen when $\lambda\,\rho < 1$

Initial leak rate λ *out of* $m=0$ *Modal density* ρ *in* $m=0$

Value of $\lambda\,\rho$ for these measurements ~ 0.04 to 0.25

Summary (re diffuse waves in axisymmetric body)

After transient source acts at center:

Energy is initially all in axisymmetric waves, $m = 0$
So energy is concentrated at center by factors~125

Energy then slowly leaks to other waves $m \neq 0$
due to scattering by axisymmetry breaking features
causes energy to be less concentrated at center

Migration to other m soon ceases
due to Dynamical Anderson Localization
(significant when leak rate λ times modal density $\rho < 1$)
residual permanent concentration of energy at center

You will sometimes hear it said that

$$\partial C / \partial \tau \sim G$$

or

$$C \sim \partial G / \partial \tau$$

so which is it?

Answer: They are equally oversimplifications

We recall

$$\partial C / \partial \tau = -\{G - G^{\text{time reversed}}\} \textit{ convolved with } F(\tau)$$

If $F(\omega) \sim$ constant in ω, then $dC/d\tau \sim -G$ (i.e equipartition in modal energies)
If $F(\omega) \sim \omega^2$, then $C \sim dG/d\tau$ (i.e. equipartition in square modal amplitudes)

So it depends on the spectrum of your noise
If your process is very narrow band, there is essentially no difference.