Ellipticity :
theory and pipeline of matlab routines

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The mode splitting due to rotation splits the ${}_nS_l$ multiplet's original spectral peak into m singlet's peaks on both sides (corresponding to $+m$ and $-m$). When the fluid angular velocity $\Omega(r,\theta)$ is unknown, it is not obvious to identify the correct $|m|$ of each peak. For that reason, it was decided to replace the sphere of ZoRo1 by a spheroid for ZoRo2. The spheroid remains symmetric around the rotation axis, but is flattened at the poles. The meridional cross-section is elliptical. This geometric deviation from sphericity splits the reference spherical multiplets.

In my Notes_acoustic_pipeline document, I left aside the treatment of the ellipticity of the container, which is essential for the ZoRo2 experiment. The reason is that the article of Dahlen (1968) I used for first-order ellipticity contained errors, and that I have coded the second-order effects, following Mehl (2007), which require a good deal of explanations. In parallel, David (with COMSOL) and Jeremie (with XXX) have computed exact mode frequencies for spheroids, which have been essential for correcting typos in the literature[2] and bugs in my programs. I present here the theory and the routines I have written and tested.

1. Mode splitting due to ellipticity: Dahlen's theory

Since the shape of the Earth is also spheroidal because of the centrifugal acceleration in its rotating frame, the effect of 'ellipticity' has been studied by seismologists. I originally followed the derivation performed by Dahlen (1968). However, the Appendix D of the book by Dahlen and Tromp (1998) points out a series of errors in the treatment of corrections to normal modes for the Earth. In particular, the formulation of Dahlen (1968) suffers from errors that have been corrected in Dahlen (1976). It turns out that in our case, the error 'only' changes the sign of the frequency perturbation, although the formulas look terribly different!

I first recall Dahlen (1968)'s derivation, and only indicate at the end the corrected expression derived from Dahlen (1976).

1.1. Dahlen's ellipticity

Dahlen defines the ellipticity ϵ of a boundary at radius around r_o as follows:

$$r(\theta)\left[1-\frac{2}{3}\epsilon P_2^0(\cos\theta)\right]=r_o. \tag{1}$$

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[2] I have added correction tags to my sample of Mehl (2007)'s pdf.

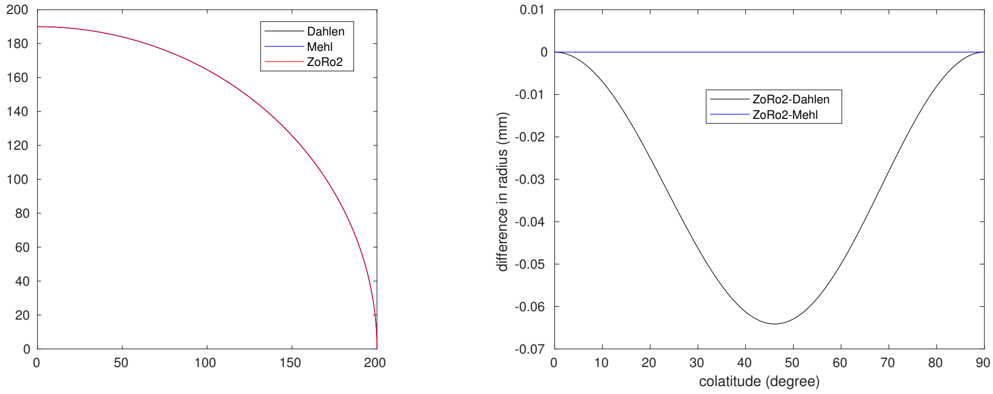

Figure 1: Comparison between the shape of ZoRo2 and that given by Dahlen's spheroid. Also shown Mehl's oblate spheroid.

Given the equatorial radius r_{eq} (for $\theta = \pi/2$) and the polar radius r_{pol} (for $\theta = 0$), one gets:

$$r_o = \frac{3}{\frac{1}{r_{pol}} + \frac{2}{r_{eq}}} \tag{2}$$

$$\epsilon = 3\,\frac{1 - \frac{r_{eq}}{r_{pol}}}{2 + \frac{r_{eq}}{r_{pol}}}. \tag{3}$$

For ZoRo2, we have: $r_{eq} = 200$ mm, $r_{pol} = 190$ mm, which yields: $r_o = 196.55$ mm and $\epsilon = -0.0517$.

Figure 1 compares the shapes of ZoRo2 and Dahlen's spheroid.

1.2. Dahlen (1968)'s treatment

Dahlen develops the ellipticity perturbation at the first order, and obtains that the frequency shift ${}_n\delta_l^m = {}_nf_l^m - {}_nf_l$ of the ${}_nS_l^m$ singlet with respect to the spherical multiplet's degenerate frequency ${}_nf_l$ varies as (Dahlen, 1968, Appendix A):

$$\frac{{}_n\delta_l^m}{{}_nf_l} = \left(-\frac{1}{3}l(l+1) + m^2\right){}_n\gamma_l\,\epsilon. \tag{4}$$

Note that the frequency of the $m = 0$ singlet ${}_nS_l^m$ of the spheroid is displaced with respect to the frequency of the ${}_nS_l$ multiplet of the corresponding $r = r_o$ sphere. Also note that, for given n and l values, the splitting due to ellipticity varies as the *square* of the azimuthal order m.

The expression of the ${}_n\gamma_l$ is derived in the Appendix A of Dahlen (1968), for the case of the Earth, which includes the effect of self-gravitation and of shear waves. I took Dahlen's expression and chopped off all terms that are not relevant in our case... Let me copy his original expression:

$${}_n(\delta\omega)_l^m\left[{}_nL_l + l(l+1)\,{}_nM_l\right] = \frac{b\,\epsilon(b)}{3\,{}_n\omega_l}\Big[\kappa\,{}_nK_l(r) + \mu\,{}_nM_l(r) + \rho_0\,{}_n\tilde{R}_l(r)\Big]_-^+, \tag{5}$$

with:

$$ {}_nK_l(r) = A_l^m (U' + F)^2 \text{ with } F = \frac{1}{r}\left[2U - l(l+1)V\right], \tag{6} $$

and where κ is the bulk modulus, μ the shear modulus, and $[]_-^+$ represents the jump of the quantity inside the bracket at the elliptical boundary. Translating into my notations: $b \equiv r_o$, $\epsilon(b) \equiv \epsilon$, $U \equiv \xi_r$, $V \equiv \xi_h$ (compare equations 7-8 of Notes_acoustic_pipelines with Dahlen's expressions p.342), ${}_n(\delta\omega)_l^m \equiv 2\pi\, {}_n\delta_l^m$, and ${}_n\omega_l \equiv 2\pi\, {}_nf_l$. The bracket on the left hand side is our normalization integral ${}_nI_l$ times ρ_f. Removing the terms related to the gravity potential that appear in ${}_n\tilde{R}_l(r)$, it remains for that term:

$$ {}_n\tilde{R}_l(r) = -{}_n\omega_l{}^2 (A_l^m \xi_r^2 + B_l^m \xi_h^2), \tag{7} $$

where

$$ A_l^m = \frac{l(l+1) - 3m^2}{(2l-1)(2l+3)} \tag{8} $$

$$ B_l^m = (l(l+1) - 3)\, A_l^m. \tag{9} $$

Hence, removing the μ-term on the right hand side of equation 5, I get:

$$ {}_n\gamma_l = \frac{r_o}{(2l-1)(2l+3)\, {}_nI_l} \left\{ \frac{1}{{}_nk_l^2} \left[\frac{d\xi_r}{dr} + \frac{1}{r_o}(2\xi_r - l(l+1)\xi_h) \right]^2 - \xi_r^2 - (l(l+1) - 3)\, \xi_h^2 \right\}, \tag{10} $$

where all quantities within the brace {} are evaluated at $r = r_o$ (*i.e.,* $r = 1$ in dimensionless radius) since $\xi_r = \xi_h = 0$ within the rigid outer shell, and where I used ${}_nk_l = {}_n\omega_l / c$ and $\kappa = \rho_f\, c^2$, where κ is the bulk modulus of the fluid.

1.3. Dahlen (1976)'s correction

As noted earlier, Dahlen and Tromp (1998) recall a long history of bugs in the literature on this topic. In particular, the treatment of ellipticity by Dahlen (1968), which I described above, contained errors that were corrected by Dahlen (1976). Equation 5 remains unchanged except that the expression of ${}_nK_l(r)$ is completely different from that of equation 6, which becomes:

$$ {}_nK_l(r) = A_l^m \left[F^2 - (U')^2 + \frac{6}{r} V (U' + F) \right], \tag{11} $$

where the latter term only comes in at a fluid-solid interface (such as in our case).
Surprisingly, the corrected formulation only results in changing the sign of ${}_n\gamma_l$ in our case.

2. Mode splitting due to ellipticity: Mehl's theory

Jim Mehl is a theoretician of the school of scientists who use 'quasi-spherical resonators' to determine with high precision the Boltzmann constant from the frequency of acoustic modes. That school has developed tools to take into account all sorts of perturbations that can affect these modes. In particular, Mehl (now 79 years old) developed over the years the theory for taking into account the shape of the container, considered as deviations from a perfect sphere.

I have been using the method that Jim Mehl published in 2007, which extends to the second-order perturbations in ellipticity. There are two remarkable elements of his theory:

1. He uses the formalism proposed by Morse and Feshbach (1953) to compute the eigenfrequencies of an acoustic cavity resonator enclosed within an unperturbed cavity.
2. He evaluates explicitly the sum on n (radial mode number) that enters these expressions, using a residue technique.

As often done by this community, he computes the relative perturbation to apply to the *square* of the k wavenumber (or f frequency), with respect to that of a perfect sphere whose volume is the same as that of the deformed container. This separates the effect of volume change from that of shape. The shape $\mathcal{F}$ is a function of θ and φ, and will be projected on spherical harmonics. Equation (2) of Mehl (2007) expresses the bounding surface of the resonator as:

$$r_s = a[1 - \epsilon\mathcal{F}(\theta,\varphi)], \tag{12}$$

where ϵ is a small positive parameter, and $\mathcal{F}$ is a smooth positive function.

Equation (21) of Mehl (2007) further decomposes $\mathcal{F}$ into two terms as:

$$\mathcal{F} = \mathcal{F}_0 + \epsilon\mathcal{F}_1 + O(\epsilon^2). \tag{13}$$

The fundamental result of Mehl (2007) is given in his equation (32), which I reproduce here:

$$\begin{aligned}\frac{(ka')^2 - {}_n\xi_l^2}{{}_n\xi_l^2} &= \frac{A_{nlmnlm}}{{}_nk_l^2 N_{nlm}} + \frac{4\epsilon^2}{{}_n\xi_l^2 - l(l+1)} \sum_{l'm'} |B^{(n)}_{lml'm'}|^2 S_{lnl'} \\ &+ \frac{2\epsilon^2 |B^{(n)}_{lm00}|^2}{{}_n\xi_l^2\left[{}_n\xi_l^2 - l(l+1)\right]} - 2\epsilon\langle\mathcal{F}_0\rangle + \epsilon^2\left[-\langle\mathcal{F}_0\rangle^2 + 2\langle\mathcal{F}_0^2\rangle - 2\langle\mathcal{F}_1\rangle\right] \\ &- 2\epsilon\langle\mathcal{F}_0\rangle \frac{A_{nlmnlm}}{{}_nk_l^2 N_{nlm}} + O(\epsilon^3),\end{aligned} \tag{14}$$

where $\langle\rangle$ denotes the average over solid angle. See Mehl (2007)'s article for a complete description of the terms. Although I have been troubled by many typos in this article, I don't think there is any problem with his equation (32).

To go further, one needs the expressions of $\frac{A_{nlmnlm}}{{}_nk_l^2 N_{nlm}}$, $|B^{(n)}_{lml'm'}|$ and $S_{lnl'}$, respectively given by equations (22), (24), and (25) of Mehl (2007).

Let me reproduce his equation (22):

$$\frac{A_{nlmnlm}}{{}_nk_l^2 N_{nlm}} = 2\epsilon \frac{\int \left[{}_n\xi_l^2\, |Y_l^m|^2 \left(\mathcal{F}_0 + \epsilon\mathcal{F}_1 - \epsilon\mathcal{F}_0^2\right) - |r\nabla Y_l^m|^2 \left(\mathcal{F}_0 + \epsilon\mathcal{F}_1\right)\right] d\Omega}{{}_n\xi_l^2 - l(l+1) - 2\epsilon \int {}_n\xi_l^2\, |Y_l^m|^2\, \mathcal{F}_0\, d\Omega} + O(\epsilon^3), \tag{15}$$

where Y_l^m is a fully normalized spherical harmonic of degree l and order m.

His equation (24) is:

$$B^{(n)}_{lml'm'} = \int \left[{}_n\xi_l^2 \left(Y_{l'}^{m'}\right)^* Y_{lm} - \left(r\nabla Y_{l'}^{m'}\right)^* r\nabla Y_{lm}\right] \mathcal{F}_0\, d\Omega. \tag{16}$$

The $S_{lnl'}$ sums are the only elements that are linked to the radial functions, and they are evaluated explicitly by Mehl (2007). They are given by his equations (26) and (27) but they contain some typos. I correct them below:

For $l' \neq 0$:

$$S_{lnl'} = -\frac{j_{l'}({}_n\xi_l)}{2\,{}_n\xi_l\, j'_{l'}({}_n\xi_l)} \qquad \text{for } l' \neq l$$
$$S_{lnl} = \frac{{}_n\xi_l^2 - 3l(l+1)}{4\left[{}_n\xi_l^2 - l(l+1)\right]^2} \quad \text{for } l' = l \tag{17}$$

For $l' = 0$:

$$S_{ln0} = -\frac{1}{2\,{}_n\xi_l^2} - \frac{j_0({}_n\xi_l)}{2\,{}_n\xi_l\, j'_0({}_n\xi_l)} \quad \text{for } l \neq 0$$
$$S_{0n0} = -\frac{1}{4\,{}_n\xi_0^2} \qquad \text{for } l = 0 \tag{18}$$

2.1. Decomposition of $\mathcal{F}$ in spherical harmonics

The next step is to decompose the functions $\mathcal{F}_0$, $\mathcal{F}_1$, and $\mathcal{F}_0^2$ in spherical harmonics, such as:

$$\mathcal{F} = \sum_{\lambda\mu} F_\lambda^\mu Y_\lambda^\mu. \tag{19}$$

For an 'oblate spheroid' as defined by Mehl, the usual ellipse equation:

$$\left(\frac{x}{r_{equ}}\right)^2 + \left(\frac{y}{r_{pol}}\right)^2 = 1 \tag{20}$$

can be written as (equation (53) of Mehl (2007)):

$$r_s = \frac{r_{equ}}{\sqrt{1 + (2\epsilon + \epsilon^2)\cos^2\theta}} = r_{equ}(1 - \epsilon\mathcal{F}) = r_{equ}[1 - \epsilon(\mathcal{F}_0 + \epsilon\mathcal{F}_1)], \tag{21}$$

where the small parameter ϵ is the ellipticity defined by:

$$r_{pol} = \frac{r_{equ}}{1 + \epsilon}. \tag{22}$$

One deduces Mehl's equation (54) and the following (fully normalized) spherical harmonic coefficients for $\mathcal{F}_0$, $\mathcal{F}_1$ and $\mathcal{F}_0^2$:

$$\mathcal{F}_0 = \frac{2}{3}\sqrt{\pi}\, Y_0^0 + \frac{4}{3}\sqrt{\frac{\pi}{5}}\, Y_2^0$$
$$\mathcal{F}_1 = -\frac{4}{15}\sqrt{\pi}\, Y_0^0 - \frac{22}{21}\sqrt{\frac{\pi}{5}}\, Y_2^0 - \frac{8}{35}\sqrt{\pi}\, Y_4^0 \tag{23}$$
$$\mathcal{F}_0^2 = \frac{2}{5}\sqrt{\pi}\, Y_0^0 + \frac{8}{7}\sqrt{\frac{\pi}{5}}\, Y_2^0 + \frac{16}{105}\sqrt{\pi}\, Y_4^0.$$

The averages over angle $\langle\mathcal{F}\rangle = \frac{1}{2\sqrt{\pi}} F_0^0$ are given by equation (55) of Mehl (2007), after correcting his value for $\langle\mathcal{F}_1\rangle$:

$$\langle\mathcal{F}_0\rangle = \frac{1}{3}, \quad \langle\mathcal{F}_1\rangle = -\frac{2}{15}, \quad \langle\mathcal{F}_0^2\rangle = \frac{1}{5}. \tag{24}$$

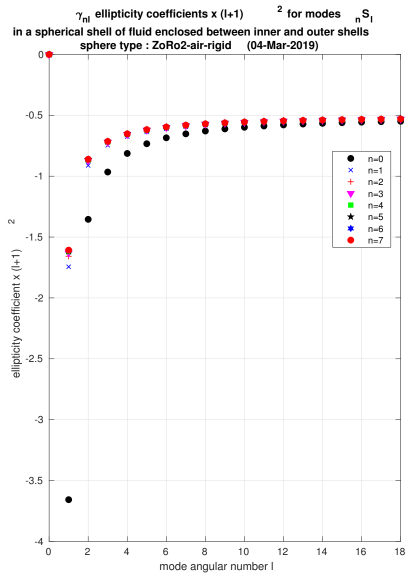

Figure 2: Ellipticity coefficients $_n\gamma_l$ (times $(l+1)^2$) of degenerate $_nS_l$ multiplets as a function of angular degree l and radial degree n in the ZoRo2 experiment filled with air, assuming a rigid outer wall, from Mehl (2007)'s first-order theory.

2.2. *Gaunt integrals*

Once the shape functions are decomposed in spherical harmonics, all integrals in equations 15 and 16 become sums (over λ and μ) of Gaunt type integrals.

Gaunt integrals are integrals of the product of three spherical harmonics such as $\int \left(Y_{l_1}^{m_1}\right)^* Y_{l_2}^{m_2} Y_{l_3}^{m_3} d\Omega$. The integrals involving $r\nabla Y_{lm}$ terms can be linked to Gaunt integrals. I call them 'alt_Gaunt' in my programs. The relation is given by equation (35) of Mehl (2007), but a factor -1 is missing.

At this stage, I diverge a bit from Mehl (2007)'s strategy. Instead of figuring out what are the non-vanishing terms in the summations, and using symbolic algebra to express them, I let Wigner 3j symbols do the job (see Dahlen and Tromp (1998) p. 197). Note that all 'Gaunt terms' are multiplied by $_n\xi_l^2$, while 'alt_Gaunt' terms do not depend on $_n\xi_l$, and therefore are the same for all n.

Let's now isolate the ϵ- and ϵ^2-terms in equation 14.

2.3. *first-order terms*

The first-order term $A1_{Mehl}$ is rather simple: it only comes from the linear part of the A_{nlmnlm} term, with the corrective term $-2\langle\mathcal{F}_0\rangle$ due to the choice of reference sphere, where the triangle brackets $\langle\rangle$ indicate an average over solid angle. Thus:

$$_ndk1_l^m = 2\frac{\int \left[{}_n\xi_l^2 \, |Y_l^m|^2 \, \mathcal{F}_0 - |r\nabla Y_l^m|^2 \, \mathcal{F}_0 \right] d\Omega}{_n\xi_l^2 - l(l+1)} - 2\langle\mathcal{F}_0\rangle. \tag{25}$$

I define $Ak_l^m(\mathcal{F})$ and $A0_l^m(\mathcal{F})$, where $\mathcal{F}$ can be $\mathcal{F}_0$, $\mathcal{F}_1$, etc:

$$\begin{aligned} Ak_l^m(\mathcal{F}) &= 2\int |Y_l^m|^2\,\mathcal{F}\,d\Omega = 2\int\sum_\lambda |Y_l^m|^2 F_\lambda^0\,Y_\lambda^0\,d\Omega \\ A0_l^m(\mathcal{F}) &= -2\int |r\nabla Y_l^m|^2\,\mathcal{F}\,d\Omega = -2\int\sum_\lambda |r\nabla Y_l^m|^2 F_\lambda^0\,Y_\lambda^0\,d\Omega, \end{aligned} \tag{26}$$

Then:

$$_n dk1_l^m = \frac{_n\xi_l^2\,Ak_l^m(\mathcal{F}_0) + A0_l^m(\mathcal{F}_0)}{_n\xi_l^2 - l(l+1)} - 2\,\langle\mathcal{F}_0\rangle. \tag{27}$$

We compute $_n dk1_l^m$ for $m = 0$ only[3], and get the relative frequency shift as in Dahlen by:

$$\frac{_n\delta_l^m}{_n f_l} = \left(-\frac{1}{3}l(l+1) + m^2\right){_n\gamma_l}\,\epsilon, \tag{28}$$

with:

$$_n\gamma_l = -\frac{3}{2\,l(l+1)}\,{_n dk1_l^0}, \tag{29}$$

where the division by 2 comes from transforming the k^2-perturbation to a k- or f-perturbation.

Figure 2 shows the ellipticity coefficients $_n\gamma_l$ thus computed for the ZoRo2 experiment. We retrieve the same values as with Dahlen's theory.

2.4. second-order terms

We get back to equation 14 to retrieve the second order terms. The $B_{lml'm'}^{(n)}$ terms are obviously second-order, as well as the $\left[-\langle\mathcal{F}_0\rangle^2 + 2\langle\mathcal{F}_0^2\rangle - 2\langle\mathcal{F}_1\rangle\right]$ term. But we also need the second order part of the $\frac{A_{nlmnlm}}{_n k_l^2 N_{nlm}}$ term, and its first order part multiplied by $-2\epsilon\langle\mathcal{F}_0\rangle$.

All these terms involve Gaunt or 'alt_Gaunt' integrals. Using the Ak and $A0$ functions introduced in equation 26, I write the $\frac{A_{nlmnlm}}{_n k_l^2 N_{nlm}}$ term of equation 15 as:

$$\begin{aligned} \frac{A_{nlmnlm}}{_n k_l^2 N_{nlm}} &= \epsilon\,\frac{_n\xi_l^2 Ak_l^m(\mathcal{F}_0) + A0_l^m(\mathcal{F}_0)}{_n\xi_l^2 - l(l+1)} \\ &+ \epsilon^2\left[\frac{_n\xi_l^2 Ak_l^m(\mathcal{F}_1 - \mathcal{F}_0^2) + A0_l^m(\mathcal{F}_1)}{_n\xi_l^2 - l(l+1)} + \frac{_n\xi_l^2 Ak_l^m(\mathcal{F}_0) + A0_l^m(\mathcal{F}_0)}{_n\xi_l^2 - l(l+1)} \times \frac{_n\xi_l^2 Ak_l^m(\mathcal{F}_0)}{_n\xi_l^2 - l(l+1)}\right] + O(\epsilon^3). \end{aligned} \tag{30}$$

This provides the two second-order terms due to $\frac{A_{nlmnlm}}{_n k_l^2 N_{nlm}}$. The $B_{lml'm'}^{(n)}$ terms involve a sum over l' and m'. For an axisymmetric container, only modes for which $m' = m$ will contribute, because of rule $m' = \mu + m$ (equation (37) of Mehl (2007)). The sum on l' is carried out from $l' = 0$ up to $l' = l + max(\lambda)$ (note that $l' = 0$ must be included in the sum, even though the $B_{lm00}^{(n)}$ term is added next). The Wigner 3j symbols used to compute Gaunt's integrals take care of cancelling terms with l' that do not obey to the triangular rule (equations (36) and (38) of Mehl (2007)).

[3] Here we loose a bit in generality. We could instead compute the first order term for all m irrespective of the m-variation.

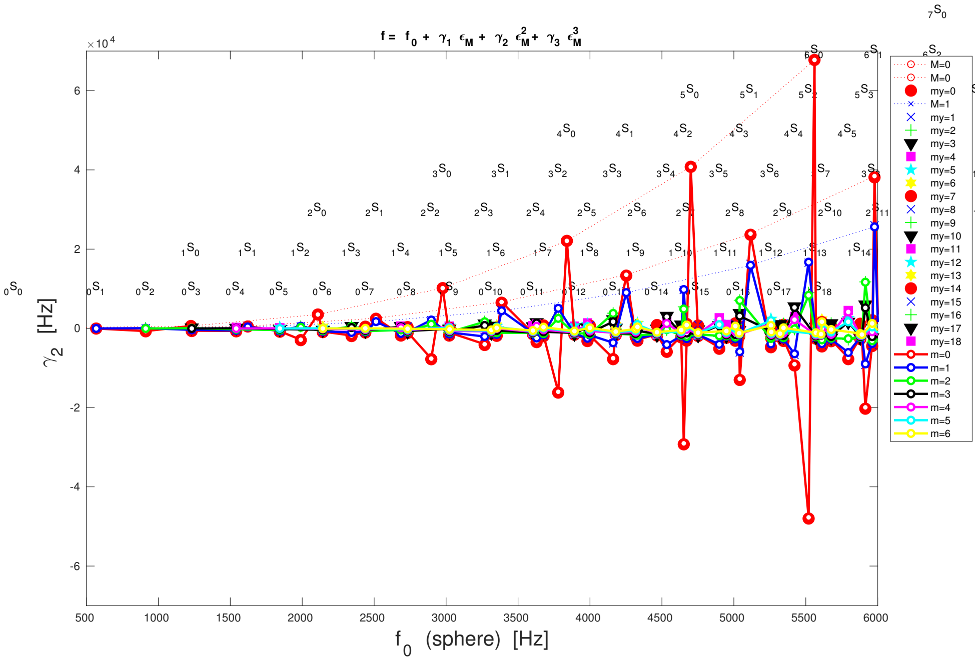

Figure 3: Second-order frequency shift γ_2 computed by David (open symbols joined by thick lines) are compared to those computed with my programs (filled symbols). Dotted lines and open circles give the results of Mehl (2007). I had to correct the exponent at the denominator of the second order term of his equation (57) from 2 to 3 to get the right fit.

2.5. *Comparison with David's results*

I could compare the second-order terms computed by my programs with the equivalent terms computed by David with COMSOL. David fitted a degree 3 polynomial to the frequency-ellipticity curve of each acoustic mode. He then expresses the mode frequency f at a given ellipticity ϵ_M (using here Mehl's definition for ellipticity) as:

$$f = f_0 + \gamma_1 \epsilon_M + \gamma_2 \epsilon_M^2 + \gamma_3 \epsilon_M^3. \tag{31}$$

The reference mode frequency f_0 is the frequency of the degenerate multiplet in a sphere of unchanged radius r_{equ}. For that reason, some conversion is needed to go from my ${}_n dk2_l^m$ to γ_2. Conversion tools are described in Appendix A.2. Figure 3 compares my results with those of David. The agreement is very good. Note that the second-order terms are large and positive for all ${}_nS_0$ modes, while they are large and negative for ${}_nS_2^0$ modes. The latter explains why the frequency of the ${}_2S_2^0$ mode is *smaller* than that of the ${}_2S_2^1$ mode in ZoRo2.

We can plot the ${}_n dk2_l^m$ fractional difference of Mehl (2007) as a function of degree l for various n and m. This is shown in Figure 4 (left) for $n = 2$ and all m and in Figure 4 (right) for $m = 2$ and all n. In both cases, I multiplied ${}_n dk2_l^m$ by $\frac{l+1}{(m+1)(n+1)^2}$, which tends to collapse all curves in a narrow range.

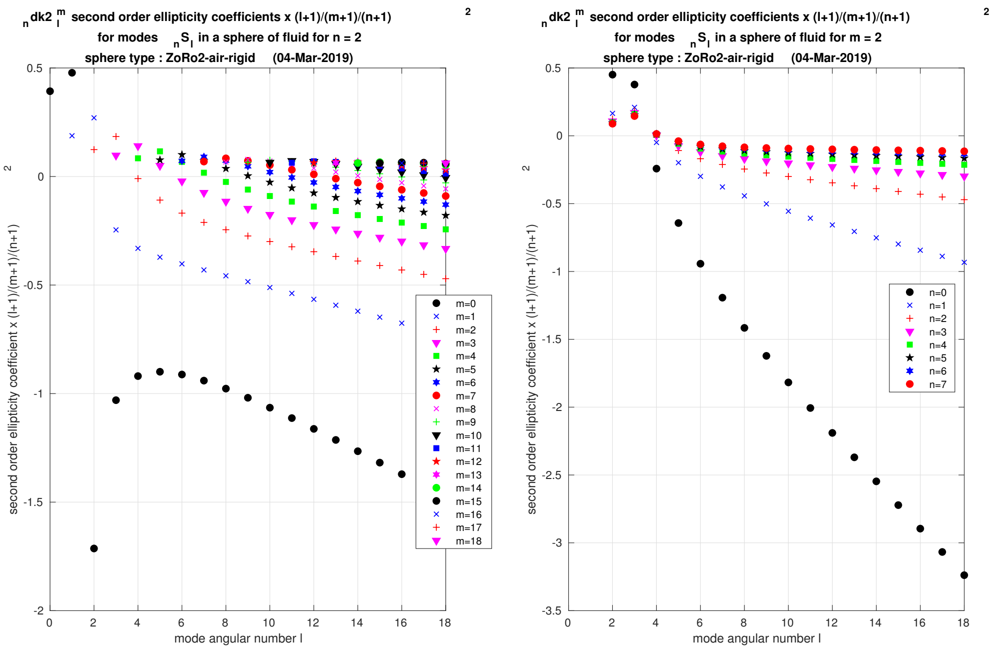

Figure 4: ${}_n dk2_l^m$ multiplied by $\frac{l+1}{(m+1)(n+1)^2}$ as a function of l : (left) for $n = 2$ and all m. (right) for $m = 2$ and all n.

3. Application to ZoRo2

Let's see how second-order perturbations in ellipticity improve our prediction of mode frequencies in ZoRo2. Figure 5 compares our new synthetic spectrum to an actual ZoRo2 spectrum. In this zoom, note that the frequency of the ${}_2S_2^0$ is clearly smaller than that of the ${}_2S_2^1$ mode. This is only possible thanks to the second-order effect. Unfortunately, the $m = 0$ peaks seem to be absent from the experimental spectrum...

4. matlab programs

The matlab scripts 'compute_modes_menu.m' and 'draw_elliptic_rotation_spectra.m' are described in the 'Notes_acoustic_pipeline' document. Most relevant functions are placed in the 'ellipticity_library' folder.

4.1. matlab functions

- ellipticity_coefficient_1976.m: computes the first-order coefficient ${}_n\gamma_l$ needed to calculate the splitting of degenerate ${}_nS_l$ modes due to ellipticity, according to Dahlen (1976), adapted to acoustic modes. Can be used for shells with an inner sphere and an elliptical outer shell, with ellipticity defined as in Dahlen (1968), which differs slightly from the ellipticity of ZoRo2.

- ellipticity_coefficient_Mehl.m: computes the first-order coefficient ${}_n\gamma_l$ needed to calculate the splitting of degenerate ${}_nS_l$ modes due to ellipticity, according to Mehl (2007), adapted to acoustic modes. Not coded yet for the presence of an inner sphere. To be used with ellipticity defined as in Mehl (2007) for an oblate spheroid,

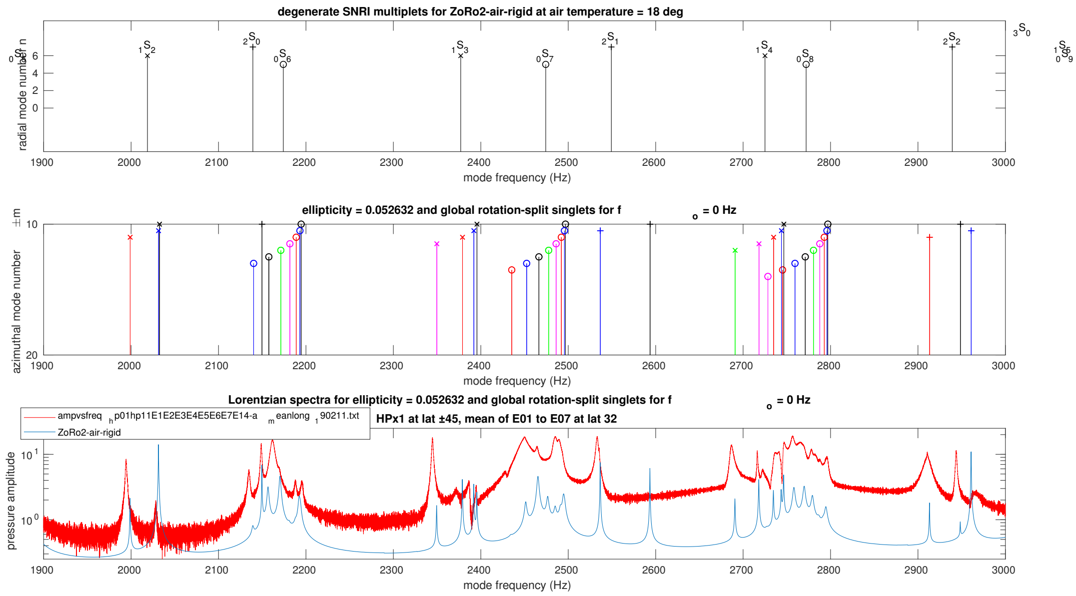

Figure 5: Comparison between a synthetic spectrum for ZoRo2 using the second-order perturbation theory of Mehl (2007) with an actual spectrum obtained by averaging over the 7 longitudes of the microphones (for that reason, I cancelled the longitudinal factor for the synthetic amplitudes).

which is the shape of ZoRo2. Note that this term describes the relative perturbation of k with respect to the degenerate mode in a sphere of identical volume as the spheroid.

- ellipticity_coefficient_Mehl_second_order: computes the second-order coefficient ${}_n dk2_l^m$ needed to calculate the splitting of degenerate ${}_nS_l$ modes due to ellipticity, according to Mehl (2007), adapted to acoustic modes. Not coded yet for the presence of an inner sphere. To be used with ellipticity defined as in Mehl (2007) for an oblate spheroid, which is the shape of ZoRo2. Note that this term describes the relative perturbation of the *square* of k with respect to the degenerate mode in a sphere of identical volume as the spheroid.

- Mehl_apply_second_order.m: combines the first-order ${}_n\gamma_l$ coefficient and the second-order ${}_n dk2_l^m$ coefficient to produce the total relative frequency shift of mode ${}_nS_l^m$ from the frequency of the degenerate mode ${}_nS_l$ computed for a sphere of identical volume as the spheroid (Mehl (2007)'s choice).

4.2. *Drawing functions*

- draw_ellipticity_coefficients.m: plots the ellipticity coefficients ${}_n\gamma_l$ of the ${}_nS_l$ modes as a function of angular degree l for all available n.

- draw_ellipticity_coefficients_compens.m: same as above, with ${}_n\gamma_l$ multiplied by factor $(l+1)^2$.

- draw_ellipticity_second_order_given_m.m: plots the ellipticity coefficients ${}_n dk2_l^m$ of the ${}_nS_l$ modes as a function of angular degree l for all available n, for a given m. The coefficients are multiplied by $(l+1)/(m+1)/(n+1)^2$.

- draw_ellipticity_second_order_given_n.m: plots the ellipticity coefficients ${}_n dk2_l^m$ of the ${}_nS_l$ modes as a function of angular degree l for all available m, for a given n. The coefficients are multiplied by $(l+1)/(m+1)/(n+1)^2$.

4.3. Lower level functions

- get_ellipticity: returns the reference radius r_o and ellipticity for various types of ellipticity definitions, given r_{equ} and r_{pol}.

- get_ellipticity_F_coefficients: returns the fully normalized spherical harmonics coefficients of the F functions needed in Mehl's theory, which depend on the ellipticity of the outer boundary.

- Mehl_ANN_term_first.m: computes the linear part of the A_{NN} term of Mehl (2007) from his equations (22) applied to an oblate or prolate spheroid.

- Mehl_ANN_term_second.m: computes the second order part of the A_{NN} term of Mehl (2007) from his equations (22) applied to an oblate or prolate spheroid.

- Mehl_Snllp.m: computes the $S_{lnl'}$ coefficients of Mehl (2007), from his equations (26) and (27) (ie, the sum over the ν of his equation (25)). I changed the sign of the second term of S_{nl0}, as corrected in Mehl (2010).

- Mehl_Blmlpmp_term.m: computes the $B_{lml'm'}$ term of Mehl (2007) from his equations (24) applied to an oblate (or prolate) spheroid.

- Gaunt_integral.m: computes the integral of the product of 3 fully normalized spherical harmonics $Y_{l_1}^{m_1}\, Y_{l_2}^{m_2}\, Y_{l_3}^{m_3}$ using Wigner 3j symbols. See Dahlen and Tromp (1998) p.917.

- alt_Gaunt_integral.m: computes the integral of the product of spherical harmonics angular derivatives that appear in equation (24) of Mehl (2007), which provides the $B_{lml'm'}$ coefficients. It is the integral of: $Y_{l_1}^{m_1}\,(r\nabla Y_{l_2}^{m_2})\,(r\nabla Y_{l_3}^{m_3})$ which is related to Gaunt's integral in equation (35) of Mehl (2007), in which I had to change the sign to get the right integral.

- Wigner3j.m: computes the Wigner 3j symbol using the Racah formula, written by Kobi Kraus, Technion, 2008, 2013.

4.4. Common lowest level routines

- spherical_bessel_j.m: $j_l(x)$ spherical Bessel function of degree l of the first kind.

- d_spherical_bessel_j.m: analytical derivative $dj_l(x)/dx$ of the spherical Bessel function of the first kind.

5. Conclusion

I am happy we could come up with correct first-order and second-order coefficients that provide the frequencies of singlets ${}_nS_l^m$ in our ZoRo2 experiment. There are several additional developments that would be interesting:

- extend Mehl's theory to a shell with a spherical inner sphere. It should not be difficult: simply replace the j-spherical Bessel functions by the appropriate j- and y- combination.
- extend Mehl's theory to an elastic shell.
- allow for non-axisymmetric container shape. It is possible with Mehl's formalism, but it requires an additional matrix diagonalisation and summing over m'.

Appendices

Appendix A. A few complements to better understand Mehl (2007)

Appendix A.1. from the enclosing sphere to the same volume sphere

Mehl (2007) and collaborators like referencing perturbations due to the shape of the container to the acoustic modes of a sphere of the same volume. The reason is that the first-order perturbation of the multiplet vanishes in that case, while the formalism of Morse and Feshbach (1953) they use requires that the deformed container is totaling enclosed in the reference sphere (this is also the reference David and Jeremie use). One therefore needs to go from one reference to the other.

Defining k the (dimensional) radial eigenvalue of the perturbed ${}_nS_l^m$ mode, k_N that of the degenerate ${}_nS_l$ mode in the enclosing sphere of radius a, k'_N that of that mode in the same-volume sphere of radius a', and ${}_n\xi_l$ the dimensionless eigenvalue of that same degenerate mode, one wishes to relate the 'desired fractional difference' $\frac{k^2-(k'_N)^2}{(k'_N)^2}$ to the original ratio $\frac{k^2-k_N^2}{k_N^2}$.

One has:

$$k_N a = k'_N a' = {}_n\xi_l. \tag{A.1}$$

We thus get equation (29) of Mehl (2007):

$$\frac{k^2-(k'_N)^2}{(k'_N)^2} = \frac{(ka')^2 - {}_n\xi_l^2}{{}_n\xi_l^2}. \tag{A.2}$$

Writing $\mathcal{A} = \frac{k^2-k_N^2}{k_N^2}$, we have:

$$(ka)^2 = \mathcal{A}_n\,\xi_l^2 + {}_n\xi_l^2, \tag{A.3}$$

while

$$\frac{(ka')^2 - {}_n\xi_l^2}{{}_n\xi_l^2} = \frac{\left(\frac{a'}{a}\right)^2 (ka)^2 - {}_n\xi_l^2}{{}_n\xi_l^2} = \frac{\left(\frac{a'}{a}\right)^2 \left[\mathcal{A}\,{}_n\xi_l^2 + {}_n\xi_l^2\right] - {}_n\xi_l^2}{{}_n\xi_l^2} = \left(\frac{a'}{a}\right)^2 [\mathcal{A}+1] - 1. \tag{A.4}$$

We therefore need to relate $\left(\frac{a'}{a}\right)^2$ to the shape parameters $\mathcal{F}_0$ and $\mathcal{F}_1$, to order ϵ^2. We first evaluate the volume ratio, which is equal to $\left(\frac{a'}{a}\right)^3$. Since the bounding surface of the quasi-spherical container is defined by $r_s = a\left[1 - \epsilon\mathcal{F}(\theta, \varphi)\right]$ (equation (3) of Mehl (2007)), this volume ratio is:

$$\left(\frac{a'}{a}\right)^3 = \frac{1}{4\pi}\int(1 - \epsilon\mathcal{F})^3 d\Omega. \tag{A.5}$$

We get by developing and retaining up to order 2:

$$\left(\frac{a'}{a}\right)^3 = \frac{1}{4\pi}\int\left[1 - 3\epsilon\mathcal{F} + 3\epsilon^2\mathcal{F}^2 + O(\epsilon^3)\right]d\Omega = 1 - 3\epsilon\langle\mathcal{F}\rangle + 3\epsilon^2\langle\mathcal{F}^2\rangle + O(\epsilon^3), \tag{A.6}$$

where $\langle\rangle$ denotes the average over solid angle.

Recalling equation (21) of Mehl (2007), which defines $\mathcal{F}_0$ and $\mathcal{F}_1$:

$$\mathcal{F} = \mathcal{F}_0 + \epsilon\mathcal{F}_1 + O(\epsilon^2), \tag{A.7}$$

we get:

$$\langle\mathcal{F}\rangle = \langle\mathcal{F}_0\rangle + \epsilon\langle\mathcal{F}_1\rangle, \tag{A.8}$$

and

$$\langle\mathcal{F}^2\rangle = \langle\mathcal{F}_0^2\rangle + O(\epsilon). \tag{A.9}$$

Thus:

$$\left(\frac{a'}{a}\right)^3 = 1 - 3\epsilon\langle\mathcal{F}_0\rangle - 3\epsilon^2\langle\mathcal{F}_1\rangle + 3\epsilon^2\langle\mathcal{F}_0^2\rangle + O(\epsilon^3), \tag{A.10}$$

which is equation (30) of Mehl (2007).

We then write $\left(\frac{a'}{a}\right)^2 = \left[\left(\frac{a'}{a}\right)^3\right]^{2/3}$ and use Taylor's expansion to obtain:

$$\left(\frac{a'}{a}\right)^2 = 1 - 2\epsilon\langle\mathcal{F}_0\rangle + \epsilon^2\left[-\langle\mathcal{F}_0\rangle^2 + 2\langle\mathcal{F}_0^2 - \mathcal{F}_1\rangle\right] + O(\epsilon^3), \tag{A.11}$$

in agreement with equation (31) of Mehl (2007).

It then only remains to combine equations (29) and (31) with equations (22) and (28), which give the two pieces of equation (16), to obtain equation (32), which is the base of the programs I wrote.

Appendix A.2. A few conversion tools

Based on the previous appendix, we can derive a few useful tools to convert first-order and second-order elliptic perturbations from Mehl's formalism to David or Jeremie computations.

First-order term:

$$First_{Mehl} = First_{David} - 2\langle\mathcal{F}_0\rangle. \tag{A.12}$$

Second-order term:

$$Second_{Mehl} = Second_{David} - 2\langle\mathcal{F}_0\rangle First_{David} - \langle\mathcal{F}_0\rangle^2 + 2\langle\mathcal{F}_0^2\rangle - 2\langle\mathcal{F}_1\rangle. \tag{A.13}$$

These relations are for ϵ- and ϵ^2- perturbations of $\left(\frac{{}_nk_l^m}{{}_nk_l}\right)^2 - 1$. One can convert into ϵ- and ϵ^2- perturbations of $\frac{{}_nf_l^m}{{}_nf_l} - 1$ by:

$$\frac{{}_nf_l^m}{{}_nf_l} - 1 = \frac{First}{2}\,\epsilon + \frac{1}{2}(Second - \frac{First^2}{4})\,\epsilon^2, \tag{A.14}$$

using Taylor's expansion to order 2.

Finally, the ellipticity is not defined exactly the same in Mehl (2007) and by David. Mehl defines ϵ_{Mehl} by $r_{pol} = r_{equ}/(1+\epsilon_{Mehl})$, while David has: $r_{pol} = r_{equ}(1-\epsilon_{David})$. The relation between the two is thus:

$$\epsilon_{Mehl} = \frac{\epsilon_{David}}{1-\epsilon_{David}} \text{ and } \epsilon_{David} = \frac{\epsilon_{Mehl}}{1+\epsilon_{Mehl}}. \tag{A.15}$$

Appendix A.2.1. matlab functions

These conversions are coded in functions placed in the 'checking_scripts' folder, together with a lot of scripts I have used to check the validity of my routines.

- df_David_from_Mehl_dk2.m: provides the first-order and second-order fractional frequency perturbation in David's r_{equ}-radius reference sphere from the first- and second-order k^2-fractional perturbations of Mehl's theory.
- dk2_Mehl_from_David_df.m: performs the inverse of the previous function.
- David_from_Mehl_ellipticity.m: function returning David ellipticity given Mehl ellipticity.
- Mehl_from_David_ellipticity.m: performs the inverse of the previous function.
- r_o_David_from_Mehl.m: returns the equivalent spherical radius r_o to consider for getting the mode frequency given Mehl's ellipticity and the equatorial radius.

Appendix A.3. Some useful formulas...

It is easy to bug when dealing with (fully normalized) spherical harmonics and similar stuff. One can check these decompositions on Wolfram Alpha. Here are a few formula to help figure out the right spherical harmonic coefficients for the shape functions $\mathcal{F}$.

Appendix A.3.1. oblate spheroid

$$\begin{aligned} 1 &= 2\sqrt{\pi}\,Y_0^0 \\ \cos^2\theta &= \frac{2}{3}\sqrt{\pi}\,Y_0^0 + \frac{4}{5}\sqrt{\frac{\pi}{5}}\,Y_2^0 \\ \cos^4\theta &= \frac{2}{5}\sqrt{\pi}\,Y_0^0 + \frac{8}{7}\sqrt{\frac{\pi}{5}}\,Y_2^0 + \frac{16}{105}\sqrt{\pi}\,Y_4^0. \end{aligned} \tag{A.16}$$

Appendix A.3.2. prolate spheroid

$$
\begin{aligned}
1 &= 2\sqrt{\pi}\, Y_0^0 \\
\sin^2\theta &= \frac{4}{3}\sqrt{\pi}\, Y_0^0 - \frac{4}{3}\sqrt{\frac{\pi}{5}}\, Y_2^0 \\
\sin^4\theta &= \frac{16}{15}\sqrt{\pi}\, Y_0^0 - \frac{32}{21}\sqrt{\frac{\pi}{5}}\, Y_2^0 + \frac{16}{105}\sqrt{\pi}\, Y_4^0.
\end{aligned}
\tag{A.17}
$$

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